E1 244: Detection and Estimation

Minimum Variance Unbiased Estimators

Signal model

DC level in white Gaussian noise (WGN)

$$x[n] = A + w[n], \quad n = 0, 1, \ldots, N-1$$

- A is the parameter to be estimated. Let us assume that A is *deterministic*.
- $w[n]$ is WGN having zero mean and variance σ^2

Estimator is a random variable

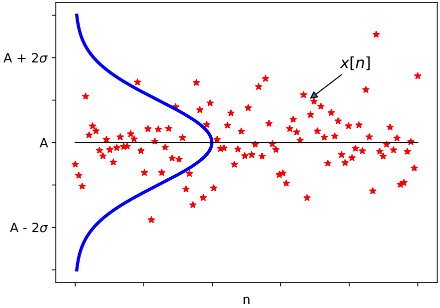
Consider two estimators

1. $\hat{A}_1 = x[0]$
2. $\hat{A}_2 = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$

Suppose $A = 1$ and

- $\hat{A}_1 = 0.97$
- $\hat{A}_2 = 0.9$

Which estimator is better?



$\hat{A}_1$ and $\hat{A}_2$ are functions of random variables. Hence the estimators are random variables!

Estimator is a random variable

- We need to study the statistical description of $\hat{A}_1$ and $\hat{A}_2$.
- Mean:

$$E[\hat{A}_1] = E[x[0]] = A$$

$$E[\hat{A}_2] = E[\frac{1}{N}\sum_{n=1}^{N} x[n]] = A$$

- Variance:

$$\text{var}[\hat{A}_1] = \sigma^2$$

$$\text{var}\left[\hat{A}_2\right] = \frac{\sigma^2}{N}$$

Although both these estimators on average attain the true value, the second estimator has a smaller variance as compared to the first estimator.

Unbiased estimator

- An estimator $\hat{\theta}$ of θ is unbiased if

$$E[\hat{\theta}] = \theta, \ \forall\theta$$

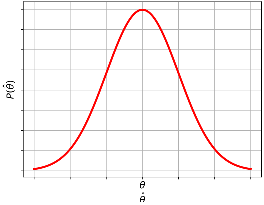
- Let $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]^T : N \times 1$ be the observations and $\hat{\theta} = g(\mathbf{x})$ with some function $g(\cdot)$, then

$$E[\hat{\theta}] = \int g(\mathbf{x}) p(\mathbf{x};\theta) d\mathbf{x} = \theta, \ \forall\theta.$$

- Here $p(\mathbf{x};\theta)$ is the data pdf parametrized by the unknown. For a fixed $\mathbf{x}$, the pdf is a function of the unknown parameter θ.

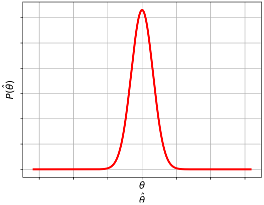
Combining unbiased estimators

- Suppose multiple estimates of the same parameter θ are available : $\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n\}$.
- A reasonable procedure to combine the estimates is, $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$. Then,
  - if all $\hat{\theta}_i$ are unbiased, then $E[\hat{\theta}] = \theta \implies \hat{\theta}$ is unbiased!
  - if $\hat{\theta}_i$ all have same variance and are uncorrelated, then $\mathrm{var}[\hat{\theta}] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}[\hat{\theta}_i] = \mathrm{var}[\hat{\theta}_1]/n$
- As $n \longrightarrow \infty$, the variance of the estimate will decrease and $\hat{\theta} \longrightarrow \theta$



$\longrightarrow$

as n increases



Biased estimators

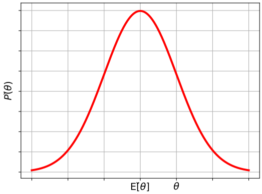
Biased estimator:

$$\mathrm{E}\left[\hat{\theta}\right] = \theta + b\left(\theta\right)$$

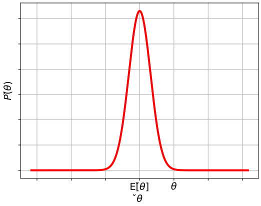
- Suppose multiple biased estimates of the parameter θ are $\{\check{\theta}_1, \check{\theta}_2, \ldots, \check{\theta}_n\}$.
- If these estimates are combined as before, then

$$\mathrm{E}\left[\check{\theta}\right] = \frac{1}{n}\sum_{i=1}^{n}\check{\theta}_i = \theta + b\left(\theta\right)$$

- As $n \longrightarrow \infty$ pdf concentrates towards a biased estimate.



$$\xrightarrow{n \longrightarrow \infty}$$



Minimum variance criterion

Optimality Criterion: Mean Squared Error

$$\begin{aligned}
\operatorname{mse}\left(\hat{\theta}\right) &= \mathrm{E}\left[\left(\hat{\theta}-\theta\right)^2\right] \\
&= \mathrm{E}\left[\left(\hat{\theta}-\mathrm{E}\left[\hat{\theta}\right]\right)+\left(\mathrm{E}\left[\hat{\theta}\right]-\theta\right)^2\right] \\
&= \operatorname{var}\left(\hat{\theta}\right)+b^2\left(\theta\right)
\end{aligned}$$

where $b\left(\theta\right)=\mathrm{E}\left[\hat{\theta}-\theta\right]$.

Note that $\operatorname{mse}\left(\hat{\theta}\right)=$ variance $+$ bias^2

Example - minimum variance criterion

- Suppose that $\check{A} = \frac{a}{N} \sum_{n=0}^{N} x[n]$
  - $\mathrm{E}\left[\check{A}\right] = aA$ and $\mathrm{var}\left(\check{A}\right) = \frac{a^2\sigma^2}{N}$
- Find 'a' that minimizes the MSE:

$$\mathrm{mse}\left(\check{A}\right) = \frac{a^2\sigma^2}{N} + (a-1)^2 A^2$$

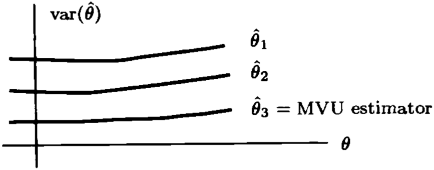
- To minimize $\mathrm{mse}\left(\check{A}\right)$, set $\frac{d}{da}\mathrm{mse}\left(\check{A}\right) = 0$ to obtain

$$a_{\mathsf{opt}} = \frac{A^2}{A^2 + \frac{\sigma^2}{N}}.$$

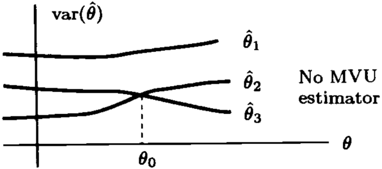
- Note that the optimal 'a' depends on the unknown parameter.
- In general, we will have unrealizable estimators ($b(\theta)$ depends on θ)
- Hence, we constrain $b(\theta) = 0$ (i.e. unbiased) and minimize the variance. Thus the name "Minimum Variance Unbiased (MVU) Estimators"

Existence of MVU estimators

Does an MVU estimator always exist?



Uniformly MVU Estimator



No MVU Estimator

MVU estimator might not always exist

Even if an MVU Estimator exists, there is no standard recipe to find it!

Example

Suppose we are given $x[0] \sim \mathcal{N}(\theta, 1)$ and $x[1] \sim \mathcal{N} \begin{cases} \mathcal{N}(\theta, 1) & \text{if } \theta \geq 0 \\ \mathcal{N}(\theta, 2) & \text{if } \theta < 0 \end{cases}$

- Define two estimators
  1. $\hat{\theta}_1 = \frac{1}{2}(x[0] + x[1])$
  2. $\hat{\theta}_2 = \frac{2}{3}(x[0] + x[1])$
- Then
  1. $\text{var}(\hat{\theta}_1) = \frac{1}{4}[\text{var}(x[0]) + \text{var}(x[1])] = \begin{cases} \frac{18}{36} & \text{if } \theta \geq 0 \\ \frac{27}{36} & \text{if } \theta < 0 \end{cases}$
  2. $\text{var}(\hat{\theta}_2) = \frac{4}{9}(\text{var}(x[0]) + \text{var}(x[1])) = \begin{cases} \frac{20}{36} & \text{if } \theta \geq 0 \\ \frac{24}{36} & \text{if } \theta < 0 \end{cases}$

Neither of these estimators have a variance uniformly less than or equal to the minima.