Cramér-Rao Lower Bound

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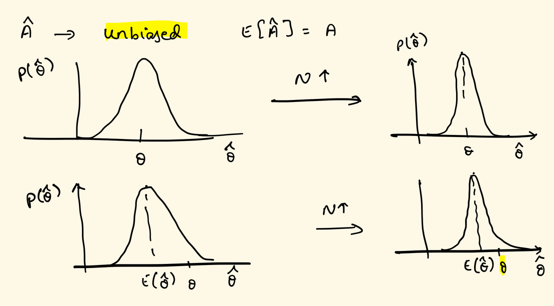
DC in WGN

$x[n] = A + w[n]$; $n = 0, 1, \ldots, N-1$ $\Big\}$ $\underline{x} = A\underline{1} + \underline{w}$

$w[n] \sim \mathcal{N}(0, \sigma^2)$

$p(\underline{x}; \theta)$

$\hat{A} \rightarrow$ unbiased $E[\hat{A}] = A$



mean square error:

$E(|\theta - \hat{\theta}|^2) = mse(\hat{\theta}) = var(\hat{\theta}) + b^2(\theta)$ ← unknown parameter

$b(\theta) = 0$

MVU estimator: (no standard recipe)

Single sample observation:

$x[0] = A + w[0]$ $w[0] \sim \mathcal{N}(0, \sigma^2)$

$E[\hat{A}] = A$

$\hat{A} = x[0]$

$var(\hat{A}) = \sigma^2$

$\hat{A}_2 = a\,x[0]$

$var(\hat{A}_2) = a^2 \sigma^2$

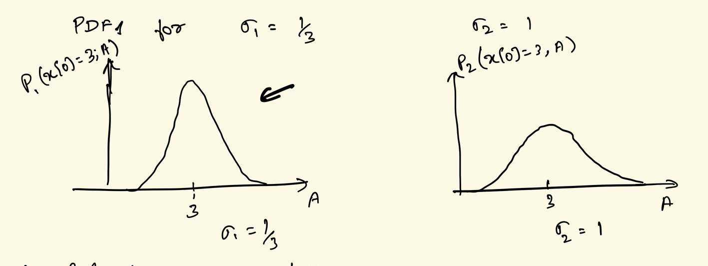
$E[\hat{A}_2] \neq A$

The accuracy of the estimator improves as σ^2 decreases

$\rightarrow p_i(x[0]; A) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{1}{2\sigma_i^2}(x[0] - A)^2\right) \leftarrow$

$x[0] = 3$

PDF1 for $\sigma_1 = \frac{1}{3}$ $\sigma_2 = 1$



values of $A > 4$ are very less less likely

$3 \pm 3\sigma = [2, 4]$ $3 \pm 3\sigma_2 = [0, 6]$

· For a fixed x, when we view PDF as a function of the unknown parameter, this function is termed as the likelihood function

Curvature of the likelihood function:

$$-\frac{\partial^2 \ln p(x[0]; A)}{\partial A^2}$$

log likelihood function:

$$\ln p(x[0]; A) = -\ln\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}(x[0] - A)^2$$

$$\frac{\partial \ln p(x[0]; A)}{\partial A} = \frac{1}{\sigma^2}(x[0] - A) \leftarrow$$

↳ How sensitive the $p(x; A)$ is to changes in A

↓ Score function $s(\underline{x}; \theta)$ ⟶

↳ scores that are close to '0' : good score

$E(s(\underline{x}; \theta)) = 0$

$-\frac{\partial^2 \ln p(x; A)}{\partial A^2} = \frac{1}{\sigma^2}$ $\Big\}$ $\sigma^2 \downarrow$ curvature increase.

$var(\hat{A}) = \sigma^2$

CRLB:

Assume that the PDF satisfies the regularity condition

(1) $E\left[\frac{\partial}{\partial\theta} \ln p(\underline{x}; \theta)\right] = 0 \quad \forall \theta$

Then, variance of any unbiased estimator $\hat{\theta}$

(2) $var(\hat{\theta}) \geq \dfrac{1}{-E\left[\frac{\partial^2 \ln p(\underline{x}; \theta)}{\partial\theta^2}\right]}$

(3) $-E\left[\frac{\partial^2 \ln p(\underline{x}; \theta)}{\partial\theta^2}\right] = E\left[\left(\frac{\partial \ln p(\underline{x}; \theta)}{\partial\theta}\right)^2\right]$

$= I(\theta)$ "Fisher information"

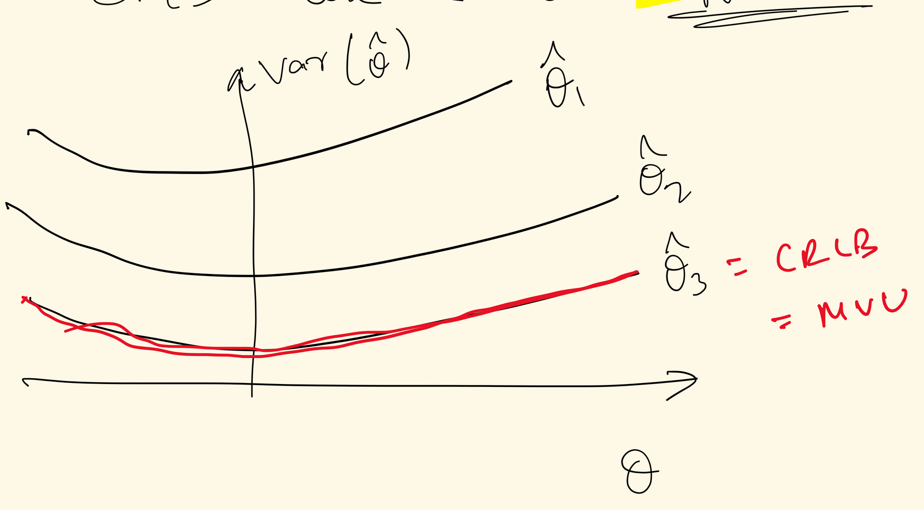
(4) $\frac{\partial \ln p(\underline{x}; \theta)}{\partial\theta} = I(\theta)[\hat{\theta} - \theta]$

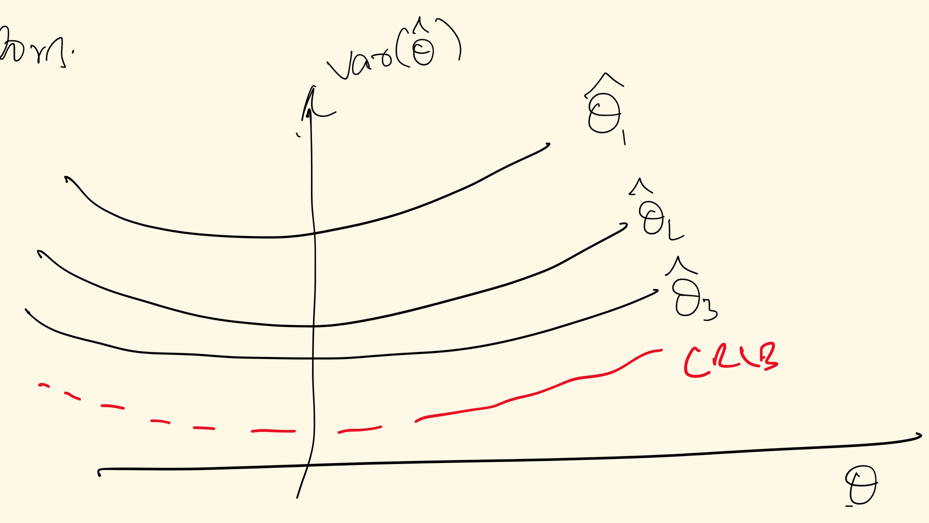
↑ $g(\underline{x}) = \hat{\theta}$

then $\hat{\theta}$ would be the MVU

$var(\hat{\theta}) = \frac{1}{I(\theta)}$.

Estimators that achieve the CRLB are called "efficient" estimators.





Regularity condition:

$$E\left[\frac{\partial}{\partial\theta} \ln p(\underline{x}; \theta)\right] = 0$$

$$\int \frac{\partial}{\partial\theta} \ln p(\underline{x}; \theta)\; p(\underline{x}; \theta)\, d\underline{x}$$

$$= \int \frac{1}{p(\underline{x}; \theta)} \frac{\partial p(\underline{x}; \theta)}{\partial\theta} \cdot p(\underline{x}; \theta)\, d\underline{x}$$

$$\overset{?}{=} \frac{\partial}{\partial\theta} \int p(\underline{x}; \theta)\, d\underline{x} = \frac{\partial}{\partial\theta} 1 = 0$$

} for cases where the integral boundary (i.e., the domain for which the PDF is non zero) doesn't depend on 'θ'

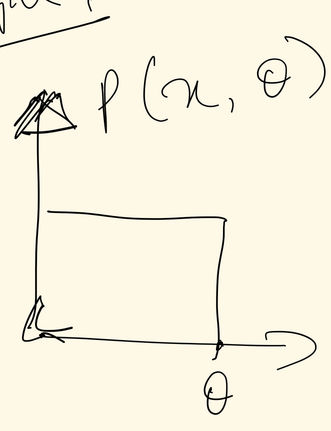
Lebnitz:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t)\, dt = \int_a^b \frac{\partial}{\partial x} f(x, t)\, dt$$

$a(x) = a$, $b(x) = b$ $\Big\}$ constants.

Example:

$p(x, \theta) = U[0, \theta]$



$$\int_0^\theta \frac{\partial}{\partial\theta} \frac{1}{\theta}\, dx \neq \frac{d}{d\theta} \int_0^\theta \frac{1}{\theta}\, dx$$

$\neq 0$

(2)

$\int (\hat{\theta} - \theta)\, p(\underline{x}, \theta)\, dx = 0$: unbiased assumption

$$\frac{\partial}{\partial\theta} \int (\hat{\theta} - \theta)\, p(\underline{x}; \theta)\, d\underline{x} = 0$$

$$-\int p(\underline{x}; \theta)\, d\underline{x} + \int (\hat{\theta} - \theta) \frac{\partial}{\partial\theta} p(\underline{x}; \theta)\, d\underline{x} = 0$$

$$\int (\hat{\theta} - \theta) \frac{\partial p(\underline{x}; \theta)}{\partial\theta}\, d\underline{x} = 1$$

$$\int (\hat{\theta} - \theta) \frac{\partial \ln p(\underline{x}; \theta)}{\partial\theta}\, p(\underline{x}; \theta)\, d\underline{x} = 1$$

$$\int (\hat{\theta} - \theta)\sqrt{p(\underline{x}; \theta)}\; \frac{\partial \ln p(\underline{x}; \theta)}{\partial\theta} \sqrt{p(\underline{x}; \theta)}\, d\underline{x} = 1 \leftarrow$$

Cauchy - Schwarz inequality:

$$\int f^2(x)\, dx \int g^2(x)\, dx \geq \left(\int f(x)\, g(x)\, dx\right)^2$$

$$\int (\hat{\theta} - \theta)^2\, p(\underline{x}, \theta)\, dx \int \left(\frac{\partial \ln p(\underline{x}, \theta)}{\partial\theta}\right)^2 p(\underline{x}; \theta)\, dx \geq 1$$

$$var(\hat{\theta}) \geq \frac{1}{E\left[\left(\frac{\partial}{\partial\theta} \ln p(\underline{x}; \theta)\right)^2\right]}$$