BLUE

Thursday, 25 March 2021 14:01

→ $\underline{x}$ observations

① → $p(\underline{x};\theta) = g(T(\underline{x}), \theta)\, h(\underline{x})$ ←

Factorize the PDF

② Is $T(\underline{x})$ complete?

$$\int_{-\infty}^{\infty} v(T)\, p(T;\theta)\, dT = 0 \quad \forall \theta$$

implies that $v(T) = 0$

③ $\left.\begin{array}{l} g(T(x)) \text{ that is unbiased} \\ E(\check{\theta} \mid T(x)) \text{ — unbiased} \end{array}\right\} = \text{MVU}$

Example:

$$x[n] = w[n] \qquad n = 0, \ldots, N-1$$

$w[n]$ is iid noise $U[0, \beta]$; $\beta > 0$

unknown parameter : $\theta = \frac{\beta}{2}$

find an MVU for θ.

$$\hat{\theta} = \bar{x} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

$E[\hat{\theta}] = \theta$ unbiased

$$\text{Var}(\hat{\theta}) = \frac{\beta^2}{12N} = \frac{1}{N} \text{var}(x[n])$$

Introduce a step function

$$u(x) = \begin{cases} 1 & \forall x > 0 \\ 0 & x < 0 \end{cases}$$

Then,

$$p(x[n];\theta) = \frac{1}{\beta}\left[u(x[n]) - u(x[n] - \beta)\right]$$

$$p(\underline{x};\theta) = \prod_{n=0}^{N-1} p(x[n];\theta)$$

$$= \frac{1}{\beta^N} \prod_{n=0}^{N-1} \left[u(x[n]) - u(x[n] - \beta)\right]$$

$$p(\underline{x};\theta) = \begin{cases} 1/\beta^N & 0 < x[n] < \beta \quad \forall n \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1/\beta^N & \max x[n] < \beta, \ \min x[n] > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\beta = 2\theta$

$$= \underbrace{\frac{1}{\beta^N}\left[u(\beta - \max x[n])\right]}_{g(T(\underline{x}), \theta)} \underbrace{u(\min x[n])}_{h(\underline{x})}$$

$T(\underline{x}) = \max x[n]$

Statistic T is order statistic

CDF $\Pr\{T \le \xi\} = \Pr\{x[0] \le \xi, \ldots, x[n] \le \xi\}$

$$= \prod_{n=0}^{N-1} Pr\{x[n] \le \xi\}$$

$$= Pr\{x[n] \le \xi\}^N$$

PDF $\frac{\partial Pr\{T \le \xi\}}{\partial \xi}$

Homework

$$p_T(\xi) = \begin{cases} 0 & \xi < 0 \\ N\left(\frac{\xi}{\beta}\right)^{N-1} \frac{1}{\beta} & 0 < \xi < \beta \\ 0 & \xi > \beta \end{cases}$$

$$\Rightarrow \quad E[T] = \frac{2N}{N+1}\theta.$$

$$T(x) = \max x[n]$$

$$\Rightarrow \quad \boxed{\hat{\theta} = \frac{N+1}{2N} \max x[n]}$$

$$E[\hat{\theta}] = \theta$$

$$\text{var}(\hat{\theta}) = \left(\frac{N+1}{2N}\right)^2 \text{var}(T)$$

$$= \frac{\beta^2}{4N(N+2)}$$

$\Rightarrow$ MVU

- optimal
- computationally (intense) → non-linear
- has a lower variance

$\frac{N+1}{2N} \cdot (\cdot)$

Variance of $\bar{x}$ → unbiased

$$\frac{\beta^2}{12N}$$

$\neq$

Linear estimator (Easy to compute)

(Suboptimal)

WGN (iid)

- linear estimator
- MVU
- optimal

Best linear unbiased estimator

(BLUE)

$$\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n] = \underline{a}^T \underline{x}$$

How to find these combining co-efficients $\{a_n\}_{n=0}^{N-1}$

- DC in Gaussian noise $\underline{a} = \frac{1}{N}\underline{1}$

- $\theta = \sigma^2$

$$x[n] = w[n]$$

MVU $\hat{\sigma}^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$ (non-linear in data)

$$\underline{y} = [x^2[0], x^2[1] \ldots —, x^2[N-1]]^T$$

$$\hat{\sigma}^2 = \underline{a}^T \underline{y}$$

$$E[\hat{\sigma}^2] = \sigma^2$$

Condition for unbiasedness:

$$E[\hat{\theta}] = E[\underline{a}^T \underline{x}] = \theta$$

$$= E\left[\sum_{n=0}^{N-1} a_n x(n)\right] = \theta$$

$$= \sum_{n=0}^{N-1} a_n \underbrace{E[x[n]]}_{} = \theta$$

I need to
Known (mean or a
scaled version
of it)

$$\sum_{n=0}^{N-1} a_n \frac{E[x[n]]}{\theta} = 1$$

$$Var(\hat{\theta}) = E\left[\left(\underline{a}^T \underline{x} - \underline{a}^T E(\underline{x})\right)^2\right]$$

$$= E\left[\left(\underline{a}^T (\underline{x} - E(\underline{x}))\right)^2\right]$$

$$= E\left[\underline{a}^T (\underline{x} - E(\underline{x}))'(\underline{x} - E(\underline{x}))\, \underline{a}\right]$$

$$= \underline{a}^T \underset{\longrightarrow \text{ Covariance of } \underline{x}}{C}\, \underline{a}$$

$C \succ 0$

"Quadratic in $\underline{a}$"

Linear model:

$$\underline{x} = \underline{h}\,\theta + \underline{w}$$

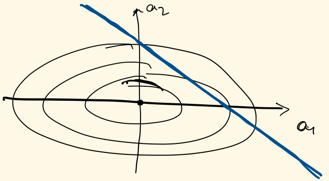
$$E[\underline{x}] = \underline{h}\,\theta$$

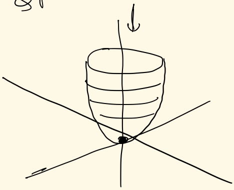
$$\underline{a}^T E[\underline{x}] = \theta$$

$$\underline{a}^T \underline{h}\,\theta = \theta$$

$$\Rightarrow \boxed{\underline{a}^T \underline{h} = 1}$$

$$\left.\begin{array}{ll} \underset{\underline{a} \in \mathbb{R}^N}{\text{minimize}} & \underline{a}^T C \underline{a} \\ \text{Subject to} & \underline{a}^T \underline{h} = 1 \end{array}\right\} \text{"Equality constrained QP"}$$





without
Constraints
the solution
is trivially $\underline{0}$

Transform the constrained problem to an unconstrained one using the method of Lagrange multipliers.

$$\mathcal{L}(\underline{a}, \lambda) = \underline{a}^T C \underline{a} + \lambda(\underline{a}^T \underline{h} - 1)$$

$$\frac{\partial \mathcal{L}}{\partial \underline{a}} = \underline{0} \in \mathbb{R}^N$$

$$= 2 C \underline{a} + \lambda \underline{h} = \underline{0}$$

$$\underline{a}(\lambda) = -\frac{\lambda}{2} C^{-1} \underline{h} \Leftarrow$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \in \mathbb{R}$$

$$\underline{a}^T \underline{h} = 1$$

$$-\frac{\lambda}{2} \underline{h}^T C^{-1} \underline{h} = 1$$

$$\lambda^* = -2 (\underline{h}^T C^{-1} \underline{h})^{-1}$$

$$\underline{a}_{opt} = (\underline{h}^T C^{-1} \underline{h})^{-1} C^{-1} \underline{h}$$

$$\underline{a}_{opt}^T \underline{h} = 1$$

$$[\quad] \underline{h}$$

$$\hat{\underline{\theta}} = \underline{a}_{opt}^T \underline{x}$$

$$= \left[(\underline{h}^T C^{-1} \underline{h})^{-1} \underline{h}^T C^{-1}\right] \underline{x} = \hat{\theta}_{MVU}$$

$$\mathrm{var}(\hat{\theta}) = \underline{a}_{opt}^T C \underline{a}_{opt}$$

$$= \frac{\cancel{\underline{h}^T C^{-1}}}{(\underline{h}^T C^{-1} \underline{h})} \cdot \cancel{C} \cdot \frac{\cancel{C^{-1} \underline{h}}}{\cancel{(\underline{h}^T C^{-1} \underline{h})}}$$

$$= (\underline{h}^T C^{-1} \underline{h})^{-1} \quad \left[= I^{-1}(\theta) \ (WGN)\right]$$

efficient → only true for WGN.

unbiased

Suppose we could design $\underline{h}$

$$C \underline{h} = \lambda \underline{h}$$

$$C^{-1} \underline{h} = \frac{1}{\lambda} \underline{h}$$

$$\mathrm{var}(\hat{\theta}) = \left(\underline{h}^{T} \underline{h} \cdot \frac{1}{\lambda}\right)^{-1} = \lambda \qquad (\underline{h}^{T} \underline{h} = \underline{1})$$

BLUE commutes over affine transformation.

$$\alpha = B\theta + b$$

B is known invertible matrix/ non-zero

$\underline{b}$ —″——— .

$$\hat{\alpha} = B\hat{\theta} + b$$

$$\theta = B^{-1}(\alpha - b)$$

$$\underline{x} = h\theta + w$$

$$= h B^{-1}(\alpha - b) + w$$

$$= \underline{h} B^{-1} \alpha - \underline{\underline{\underline{h} B^{-1} \underline{b}}} + w$$

$$\underline{x'} = \underline{x} + \underline{h} B^{-1} b = \underline{h} B^{-1} \alpha + \underline{w}$$