Lecture 3 E1 260 24/08/2021

- Weierstrass' theorem

- Convex sets : Defn. , half space & hyperplanes, Cones & dual cones, operations preserving convexity.

- Convex functions:
  - definition , linear lower bound
  - why important

References:

- Convex sets & functions : Boyd, Convex optimization, Chapter 2 and 3

Existence of optimal solutions:

Weierstrass' theorem:

Let X be a non empty subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ be lower semicontinuous (or closed) at all points of X. Assume the one of the following three conditions holds:

① X is compact (i.e., closed and bounded)

② X is closed and f is coercive

Then, $\exists \, \underline{x} \in X$ such that $f(\underline{x}) = \inf_{\underline{z} \in X} f(\underline{z})$.

$= f_{opt}$

Proof:

- Sequence has at least one lt. point
- that lt. point is in X
- $f(x)$ is bounded from below

① Compactness of X:

Let $\{z_k\} \subset X$ be a sequence such that

$$\lim_{k \to \infty} f(\underline{z}_k) = \inf_{\underline{z} \in X} f(\underline{z})$$

→ Since X is bounded, $\{z_k\}$ has at least one limit point $\underline{x}^*$.

→ X is closed, $\underline{x}^*$ belong to X.

→ lower semi-continuity:

$$f(\underline{x}^*) \leq \lim_{k \to \infty} f(\underline{z}_k) = \inf_{\underline{z} \in X} f(\underline{z}) \neq -\infty$$

$$\therefore \quad f(\underline{x}^*) = \inf_{\underline{z} \in X} f(\underline{z})$$

$\boxed{\underline{x}^* \text{ is the minimizer of } f \text{ over } X}$

② Closedness of $\mathcal{X}$ and f is coercive:

non-empty closed set $\Big\}$ $\text{dom}(f) \cap \mathcal{X} \neq \phi$

$\underbrace{\text{is coercive}}$: $f(\underline{x}) \to \infty$ as $\|\underline{x}\| \to \infty$

- Let $\underline{x}_0$ be an arbitrary point in $\mathcal{X}$
- Coerciveness: $\exists$ $M > 0$
  $f(\underline{x}) > f(x_0)$ for any $\underline{x}$ satisfying $\|\underline{x}\| > M$
- Any minimizer $\underline{x}^*$ of f over $\mathcal{X}$: $f(\underline{x}^*) \leq f(x_0)$
- From coerciveness: The set of minimizers of f over $\mathcal{X}$ is the same as the set of minimizers of f over the set $\mathcal{X} \cap B[0, M]$

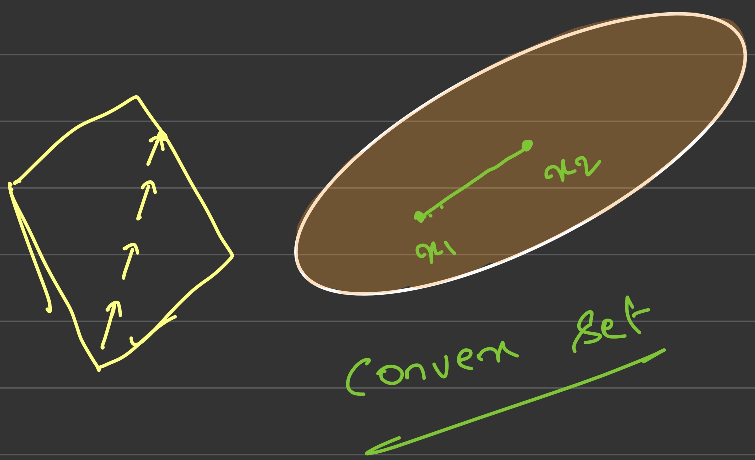
$\mathcal{X}$ closed; $\underbrace{\mathcal{X} \cap B[0,M]}_{\text{Compact}}$; $B[0,M]$ closed + bounded

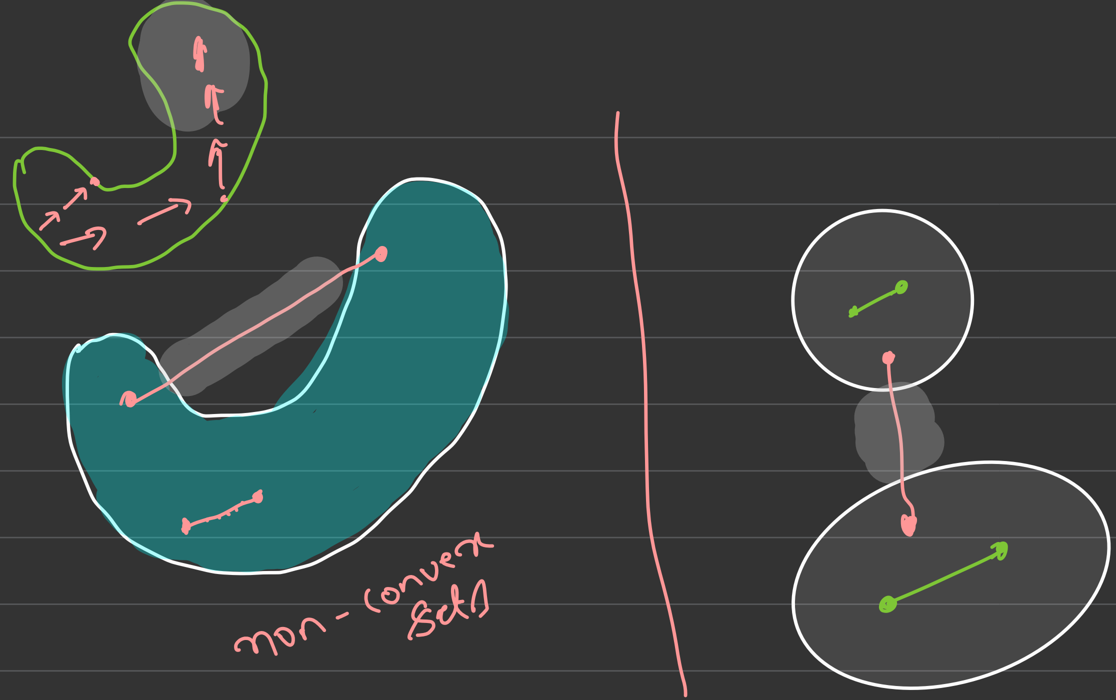
$f(x_0) \geq f(x^*)$; $\forall$ $\|\underline{x}\| \leq M$

$f(x) > f(x_0) \geq f(x^*)$; $\forall$ $\|x\| > M$

$\therefore$ from part A, $\exists$ a minimizer of f over $S \cap B[0,M]$ & over S.

Convex Sets:

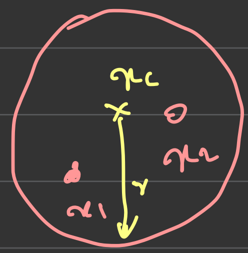


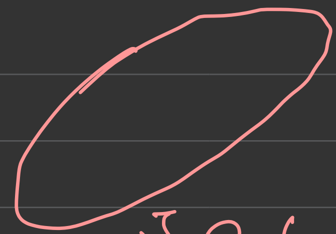


A subset C of $\mathbb{R}^n$ is convex if

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in C \, , \quad \forall \, \underline{x}_1, \underline{x}_2 \in C$$
$$\forall \, \theta \in [0,1]$$

Examples of Convex sets:





$(x - x_c)^T Q (x - x_c) \le r^2$

① Euclidean ball

$$B(\underline{x}_c, r) = \{ \underline{x} : \| \underline{x} - \underline{x}_c \| \le r)$$

$$= \{ \underline{x} : (\underline{x} - \underline{x}_c)^T (\underline{x} - \underline{x}_c) \le r^2 \}$$

$\rightarrow$ $\| \underline{x}_1 - \underline{x}_c \| \le r$ and $\| \underline{x}_2 - \underline{x}_c \| \le r$

$\rightarrow$ $\theta \in [0, 1]$

② $\mathbb{R}^n_+$ is Convex set ?

$$\| \theta \underline{x}_1 + (1-\theta) \underline{x}_2 - \underline{x}_c \|_2$$

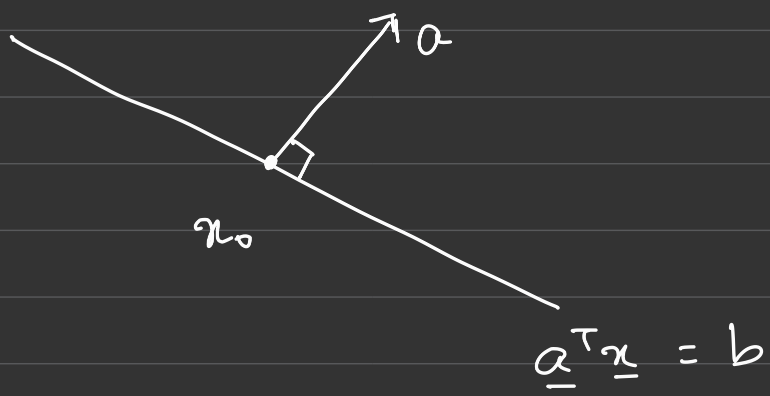
$$= \| \theta (\underline{x}_1 - \underline{x}_c) + (1-\theta)(\underline{x}_2 - \underline{x}_c) \|_2$$

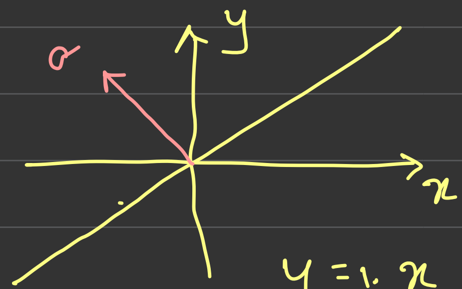
$$\le \theta \| \underline{x}_1 - \underline{x}_c \|_2 + (1-\theta) \| \underline{x}_2 - \underline{x}_c \|_2$$

$$\le r \quad \Rightarrow \quad \theta \underline{x}_1 + (1-\theta_2) \underline{x}_2 \in B(\underline{x}_c, r)$$

Hyperplanes and half spaces:

hyperplane: $\{ \underline{x} \in \mathbb{R}^n : \underline{a}^T \underline{x} = b \}$; $\underline{a} \neq 0$



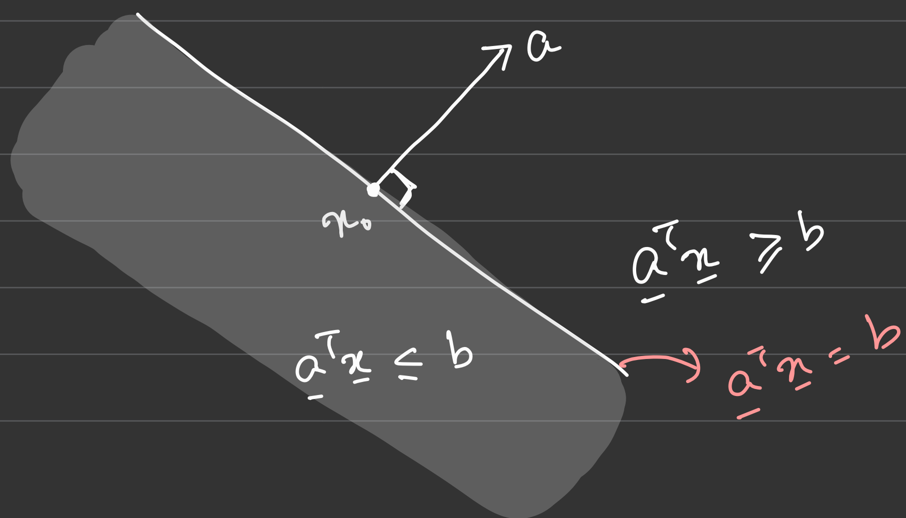


$y = 1.x + 0$

$$\underbrace{\begin{bmatrix} -1 & 1 \end{bmatrix}}_{\underline{a}^T} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\underline{a}^T \underline{x} = \underline{0}$

Half space: $H^+ = \{ \underline{x} \in \mathbb{R}^n : \underline{a}^T \underline{x} \leq b \}$



"half spaces are convex"

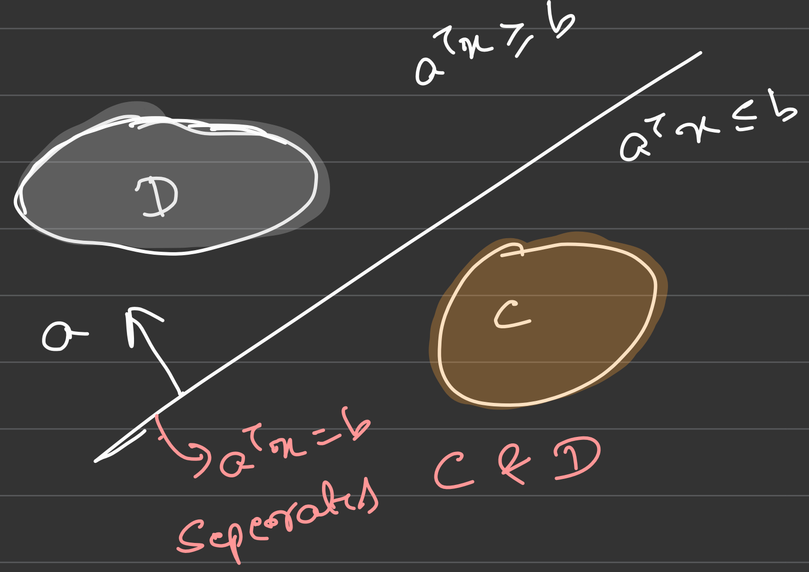
$\underline{x}_1 \in H^+$ $\underline{x}_2 \in H^+$

$\rightarrow \underline{a}^T \underline{x}_1 \leq b$; $\underline{a}^T \underline{x}_2 \leq b$

$\theta \in [0, 1]$

$a^T(\theta \underline{x}_1 + (1-\theta) x_2)$

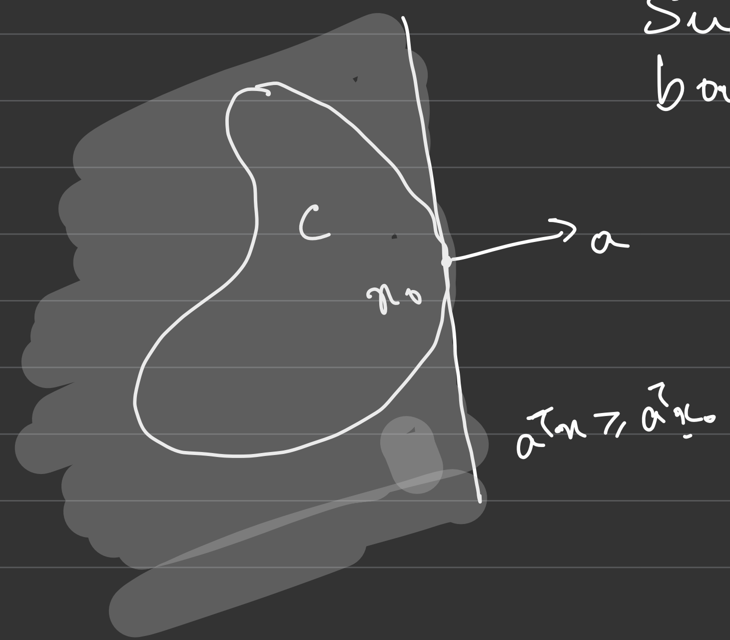
Separating and supporting hyperplane:

⇒ 

if C and D are disjoint convex sets, $\exists\ \underline{a} \neq 0,\ b$

s.t. $\underline{a}^T \underline{x} \geq b$ for $x \in D$

$\underline{a}^T \underline{x} \leq b$ for $x \in C$

⇒ 

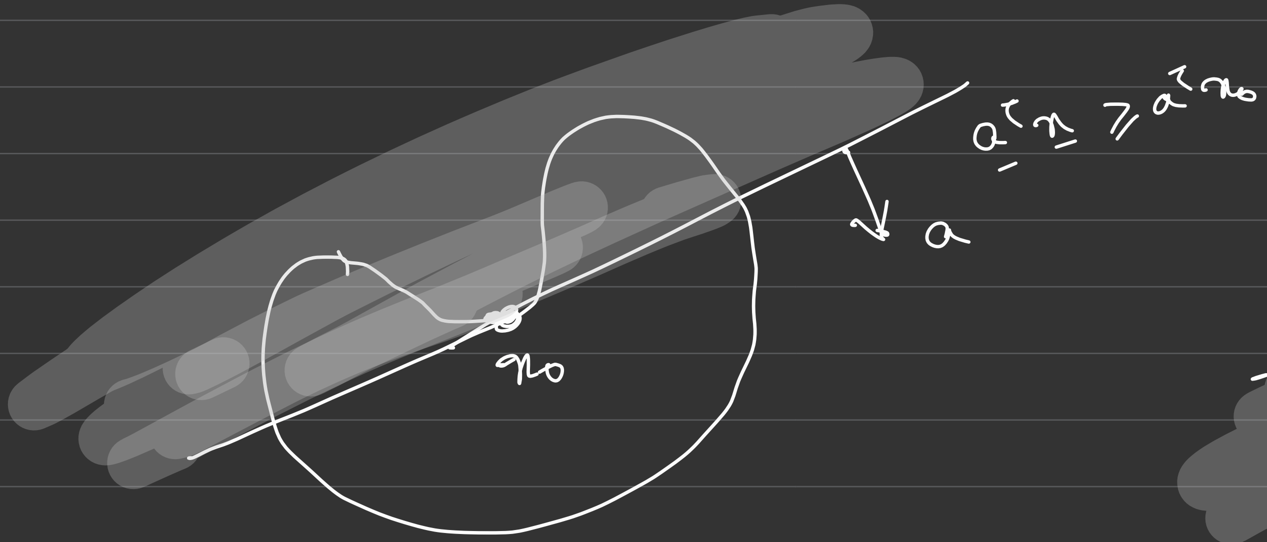
Supporting hyperplane to C at boundary point x_0

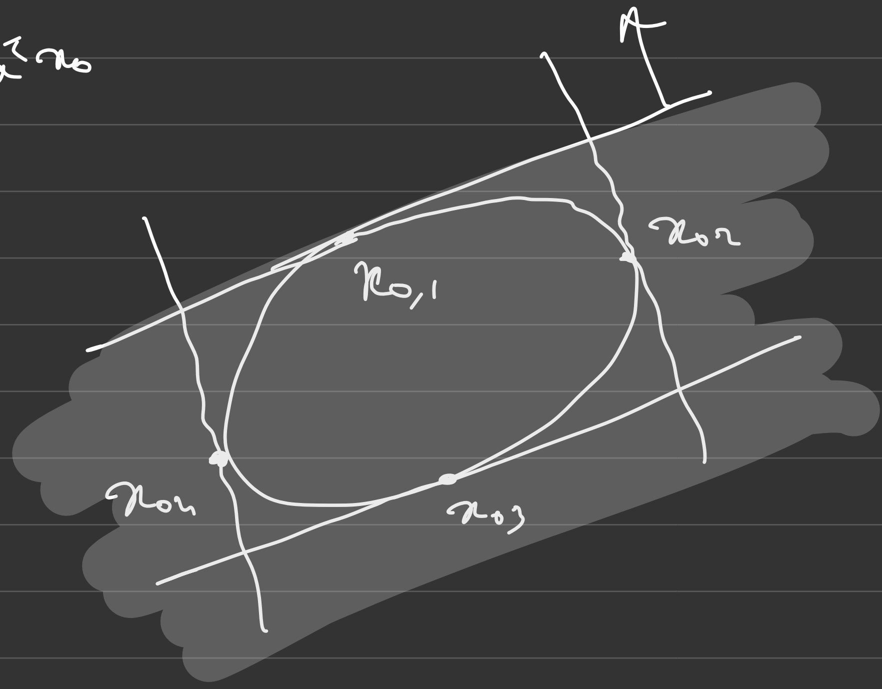
$$\{ x : \underline{a}^T \underline{x} = a^T x_0 \}$$

where $a \neq 0$ and

$$\underline{a}^T \underline{x} \leq \underline{a}^T \underline{x}_0 \quad \text{for all } x \in C$$

if C is convex, then $\exists$ a supporting hyperplane at every boundary point of C





$\rightarrow \quad \theta_1 x_1 + \theta_2 x_2 \in C$

s.t. $\quad \theta_1 + \theta_2 = 1$

$\theta_2 = 1 - \theta_1 \; ; \quad \theta_1 \in [0,1]$

Convex Cone:

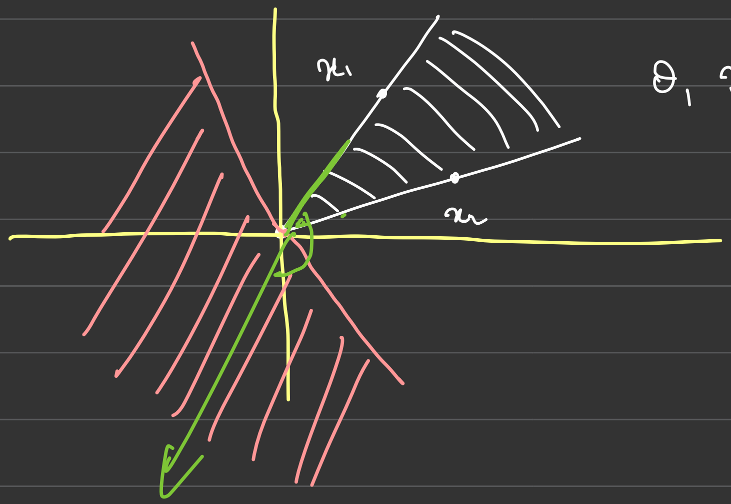
Conic non-negative combination of x_1 and x_2

$$x = \theta_1 x_1 + \theta_2 x_2 \quad \text{with } \theta_1 \geq 0, \ \theta_2 \geq 0$$

Convex Cone:

A set C is a convex cone if

$$\theta_1 x_1 + \theta_2 x_2 \in C \quad \forall x_1, x_2 \in C, \ \forall \theta_1, \theta_2 \geq 0$$



Polar cone:

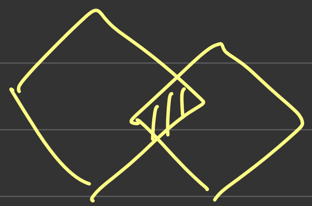
$$C^* = \{ y : x^T y \leq 0, \ \forall x \in C \}$$

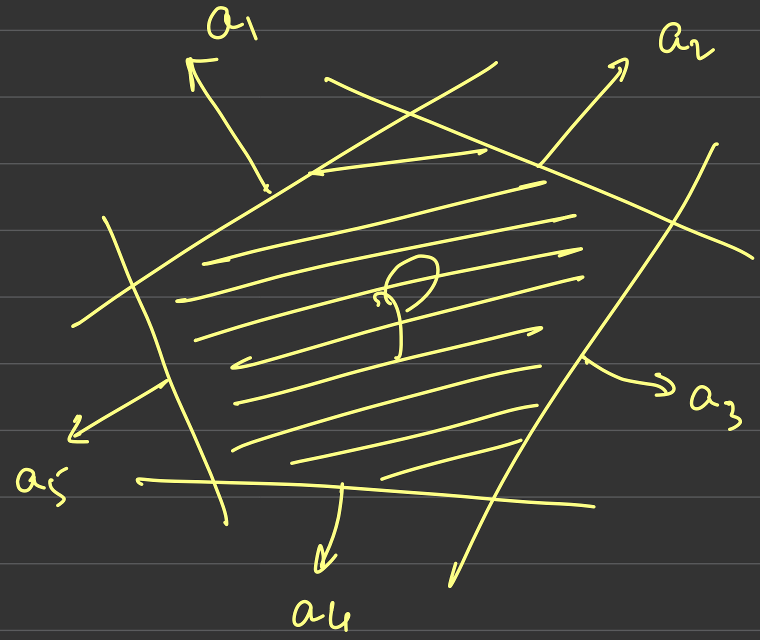
Polyhedra:

(Convex set)

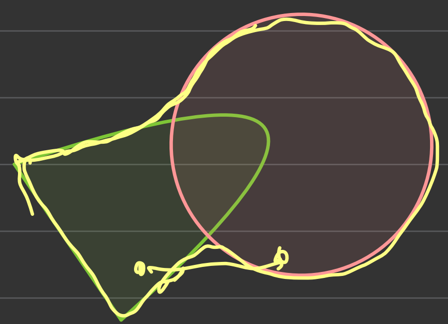
$$P = \{ \underline{x} : \underline{a}_j^T \underline{x} \le b_j \ , \ j = 1 \ldots, m$$
$$\underline{c}_j^T \underline{x} = d_j \ ; \ j = 1 \ldots p \} .$$

Bounded polyhedra is called as polytope





Intersection of finite number of half spaces & hyperplanes



Operations that preserve convexity:

- Intersection : $\bigcap_{i \in I} C_i$ of convex sets C_i

  E.g. Polytope

- Affine functions : $f : \mathbb{R}^n \to \mathbb{R}^m$ E.g. $f(\underline{x}) = A\underline{x} + \underline{b}$

$\to$ Image of a convex set $C \subseteq \mathbb{R}^n$ under f

$$f(C) : \{ f(\underline{x}) : \underline{x} \in C \}$$

$\to$ Inverse image $f^{-1}(C) : \{ x : f(x) \in C \}$

E.g. $\alpha C : \{ \alpha \underline{x} : \underline{x} \in C \}$; $\alpha + C = \{ \underline{x} + \alpha : x \in C \}$

$$C_1 + C_2 = \{ x_1 + x_2 : x_1 \in C_1 , x_2 \in C_2 \}$$

convex