1. Definitions
2. Subspaces
3. Projection
A $N$-dimensional vector is assumed to be a column vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

**Complex conjugate (Hermitian) transpose**

$$\mathbf{x}^H = (\mathbf{x}^T)^* = [x_1^*, x_2^*, \ldots, x_N^*]$$

For a discrete-time signal $x(n)$, we will use the following vectors

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \quad \mathbf{x}_n = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N+1) \end{bmatrix}$$
Matrices

- An $N \times M$ matrix has $N$ rows and $M$ columns:

\[ A = [a_{ij}] = \begin{bmatrix}
   a_{11} & a_{12} & \cdots & a_{1M} \\
   a_{21} & a_{22} & \cdots & a_{2M} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{N1} & a_{N2} & \cdots & a_{NM}
\end{bmatrix} \]

- Complex conjugate (Hermitian) transpose

\[ A^H = (A^T)^* = (A^*)^T \]

- Hermitian matrix

\[ A = A^H \]

E.g.,

\[ A = \begin{bmatrix}
   1 & 1 + j \\
   1 - j & 1
\end{bmatrix}, \quad \text{then} \quad A^H = \begin{bmatrix}
   1 & 1 + j \\
   1 - j & 1
\end{bmatrix} = A \]
Vectors

**Vector norms:** \[ \| \mathbf{x} \|_p = \left( \sum_{i=1}^{N} |x_i|^p \right)^{1/p}, \text{ for } p = 1, 2, \ldots \]

- \[ \| \mathbf{x} \|_p \geq 0 \text{ when } \mathbf{x} \neq \mathbf{0} \text{ and } \| \mathbf{x} \|_p = 0 \text{ iff } \mathbf{x} = \mathbf{0} \]
- \[ \| \alpha \mathbf{x} \|_p = \alpha \| \mathbf{x} \|_p \text{ for any scalar } \alpha \]
- \[ \| \mathbf{x} + \mathbf{y} \|_p \leq \| \mathbf{x} \|_p + \| \mathbf{y} \|_p \]

**Examples:**

**Euclidean (2-norm):** \[ \| \mathbf{x} \|_2 = \left( \sum_{i=1}^{N} x_i^* x_i \right)^{1/2} = (\mathbf{x}^H \mathbf{x})^{1/2} \]

**1-norm:** \[ \| \mathbf{x} \|_1 = \sum_{i=1}^{N} |x_i| \]

**\(\infty\)-norm:** \[ \| \mathbf{x} \|_\infty = \max_i |x_i| \]
Vectors

Inner product:

\[ \langle x, y \rangle = x^H y = \sum_{i=1}^{N} x_i^* y_i \]

- Two vectors are *orthogonal* if \( \langle x, y \rangle = 0 \); if the vectors have unit norm, then they are *orthonormal*

- Cauchy-Schwarz: \(|\langle x, y \rangle| \leq \|x\| \|y\|\)

\[ \langle x, y \rangle = \|x\| \|y\| \cos \theta \]

\( \theta \) is the angle between the two vectors. Since \(|\cos \theta| \leq 1\), the above inequality follows.

- \( 2\langle x, y \rangle \leq \|x\|^2 + \|y\|^2 \) as

\[ \|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\langle x, y \rangle \geq 0 \]
Matrices

For $A \in \mathbb{C}^{M \times N}$

- **2-norm** (spectral norm, operator norm):

  $$\|A\| := \max_x \frac{\|Ax\|}{\|x\|} \text{ or } \|A\|^2 := \max_x \frac{x^H A^H A x}{x^H x}$$

  Largest magnification that can be obtained by applying $A$ to any vector

- **Forbenius norm**

  $$\|A\|_F := \left( \sum_{i=1}^M \sum_{j=1}^N |a_{ij}|^2 \right)^{1/2}$$

  Represents energies in its entries
Linear independence, vector spaces, and basis vectors

**Linear independence**

- A collection of \( N \) vectors \( x_1, x_2, \ldots, x_N \) is called *linearly independent* if

\[
\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N = 0 \iff \alpha_1 = \alpha_2 = \cdots = \alpha_N = 0
\]

**Rank**

- The rank of \( A \) is the number of independent columns or rows of \( A \)

Prototype rank-1 matrix: \( A = ab^H \)

Prototype rank-2 matrix: \( A = ab^H + cd^H \)

- The ranks of \( A, AA^H, \) and \( A^H A \) are the same

- If \( A \) is square and full rank, there is a unique inverse \( A^{-1} \) such that

\[
AA^{-1} = A^{-1} A = I = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

- An \( N \times N \) matrix \( A \) has rank \( N \), then \( A \) is invertible \( \iff \det(A) \neq 0 \)
Subspaces

The space $\mathcal{H}$ spanned by a collection of vectors $x_1, x_2, \ldots, x_N$

$$\mathcal{H} := \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N | \alpha_i \in \mathbb{C}, \forall i\}$$

is called a *linear subspace*

- If the vectors are linearly independent they are called a *basis* for the subspace
- The number of basis vectors is called the *dimension* of the subspace
- If the vectors are orthogonal, then we have an *orthogonal basis*
- If the vectors are orthonormal, then we have an *orthonormal basis*
Fundamental subspaces of $A$

- **Range (column span) of $A \in \mathbb{C}^{M \times N}$**

  \[ \text{ran}(A) = \{Ax : x \in \mathbb{C}^N\} \subset \mathbb{C}^M \]

  The dimension of $\text{ran}(A)$ is rank of $A$, denoted by $\rho(A)$

- **Kernel (row null space) of $A \in \mathbb{C}^{M \times N}$**

  \[ \text{ker}(A) = \{x \in \mathbb{C}^N : Ax = 0\} \subset \mathbb{C}^N \]

  The dimension of $\text{ker}(A)$ is $N - \rho(A)$

- **Four fundamental subspaces**

  \[ \text{ran}(A) \oplus \text{ker}(A^H) = \mathbb{C}^M \]

  \[ \text{ran}(A^H) \oplus \text{ker}(A) = \mathbb{C}^N \]

  direct sum: $\mathcal{H}_1 \oplus \mathcal{H}_2 = \{x_1 + x_2 | x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2\}$
A square matrix $U$ is called *unitary* if $U^H U = I$ and $UU^H = I$

- Examples are rotation or reflection matrices
- $\|U\| = 1$; its rows and columns are orthonormal

A tall rectangular matrix $\hat{U}$ is called an *isometry* if $\hat{U}^H \hat{U} = I$

- Its columns are orthonormal basis of a subspace (not the complete space)
- $\|\hat{U}\| = 1$
- There is an orthogonal complement $\hat{U}^\perp$ of $\hat{U}$ such that $[\hat{U} \quad \hat{U}^\perp]$ is unitary
Projection

▶ A square matrix $P$ is a projection if $PP = P$

▶ It is an orthogonal projection if $P^H = P$
  ▶ The norm of an orthogonal projection is $\|P\| = 1$

▶ For an isometry $\hat{U}$, the matrix $P = \hat{U}\hat{U}^H$ is an orthogonal projection onto the space spanned by the columns of $\hat{U}$.

▶ Suppose $U = [\hat{U} \quad \hat{U}^\perp]_d$ is unitary. Then, from $UU^H = I_N$: 

$$\hat{U}\hat{U}^H + \hat{U}^\perp(\hat{U}^\perp)^H = I_N, \quad \hat{U}\hat{U}^H = P, \quad \hat{U}^\perp(\hat{U}^\perp)^H = P^\perp = I_N - P$$

▶ Any vector $x \in \mathbb{C}^N$ can be decomposed as $x = \hat{x} + \hat{x}^\perp$ with $\hat{x} \perp \hat{x}^\perp$:

$$\hat{x} = Px \in \text{ran}(\hat{U}) \quad \hat{x}^\perp = P^\perp x \in \text{ran}(\hat{U}^\perp)$$
Projection onto the column span of a tall matrix $A$

- Suppose $A$ has full column rank (i.e., $A^H A$ is invertible). Then,

$$P_A := A(A^H A)^{-1} A^H,$$

$$P_A^\perp := I - A(A^H A)^{-1} A^H,$$

are orthogonal projections onto $\text{ran}(A)$ and $\ker(A^H)$, respectively.

- To prove, verify that $P^2 = P$ and $P^H = P$, hence $P$ is an orthogonal projection.

- If $b \in \text{ran}(A)$, then $b = Ax$ for some $x$:

$$P_A b = A(A^H A)^{-1} A^H Ax = b$$

so that $b$ is invariant under $P_A$.

- If $b \perp \text{ran}(A)$, then $b \in \ker(A^H) \iff A^H b = 0$:

$$P_A b = A(A^H A)^{-1} A^H b = 0.$$