Deptartment of Electrical Communication Engineering
Indian Institute of Science

E9 211 Adaptive Signal Processing

1 Oct 2019, 13:30–15:00, Mid-term exam solutions

This exam has two questions (20 points). Question 2 is on the back side of this page.

A4 cheat sheet is allowed. No other materials will be allowed.

Question 1 (10 points): Temperature estimation

Let T_0 denote the initial temperature of a metal rod and assume that it is decreasing exponentially. We make two noisy measurements of the temperature of the rod at time instants t_1 and t_2 as

$$x_i = T_0 e^{-t_i} + v_i, \quad i = 1, 2,$$

where v_1 and v_2 are uncorrelated zero-mean random variables with variances $\sigma_1^2 = \sigma_2^2 = 1$, respectively.

(3pts) (a) Assume that T_0 is a constant (i.e., deterministic). Given x_1 and x_2, compute the *Best Linear Unbiased Estimator* (BLUE) $\hat{T}_0$.

(2pts) (b) Show that the estimator $\hat{T}_0$ is unbiased and give the resulting minimum mean-square error.

(3pts) (c) Now, suppose T_0 is a zero-mean random variable with variance $\sigma_T^2 > 0$. Assuming that T_0 and v_i are uncorrelated, compute the linear-least-mean squares estimator $\hat{T}_{0,\text{lmmse}}$.

(2pts) (d) Give the resulting minimum mean-square error for the estimator $\hat{T}_{0,\text{lmmse}}$ and compare it with the one obtained in Part (b) of this question with $\sigma_1^2 = \sigma_2^2 = 1$. Is $\hat{T}_{0,\text{lmmse}}$ unbiased?

Solutions

(a) The estimator $\hat{T}_0 = \mathbf{w}^H\mathbf{x}$ is computed by solving the constrained optimization problem

$$\underset{\mathbf{w}}{\text{minimize}} \quad \mathbf{w}^{\text{H}}\mathbf{R}_v\mathbf{w} \quad \text{subject to} \quad \mathbf{w}^{\text{H}}\mathbf{h} = 1$$

whose solution is

$$\mathbf{w}_{\text{opt}} = \mathbf{R}_v^{-1}\mathbf{h}(\mathbf{h}^{\text{T}}\mathbf{R}_v^{-1}\mathbf{h})^{-1} \quad \Rightarrow \hat{T}_0 = (\mathbf{h}^{\text{T}}\mathbf{R}_v^{-1}\mathbf{h})^{-1}\mathbf{h}^{\text{T}}\mathbf{R}_v^{-1}\mathbf{x}.$$

For this problem, since

$$\mathbf{R}_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{h} = \begin{bmatrix} e^{-t_1} \\ e^{-t_2} \end{bmatrix}$$

we have

$$\hat{T}_0 = \left(e^{-2t_1} + e^{-2t_2}\right)^{-1}\left(e^{-t_1}x_1 + e^{-t_2}x_2\right).$$

(b) Since $E\{\hat{T}_0\} = \left(e^{-2t_1} + e^{-2t_2}\right)^{-1} E\left\{\left(e^{-t_1}x_1 + e^{-t_2}x_2\right)\right\} = T_0$, the estimator is unbiased. Furthermore, the minimum mean-squared error is $\mathbf{w}_{\text{opt}}^{\text{H}}\mathbf{R}_v\mathbf{w}_{\text{opt}} = \left(e^{-2t_1} + e^{-2t_2}\right)^{-1}$.

(c) The LMMSE estimator is given by

$$\hat{T}_{0,\text{lmmse}} = [\sigma_T^{-2} + \mathbf{h}^{\text{T}}\mathbf{h}]^{-1}\mathbf{h}^{\text{T}}\mathbf{x} = \left(\sigma_T^{-2} + e^{-2t_1} + e^{-2t_2}\right)^{-1}\left(e^{-t_1}x_1 + e^{-t_2}x_2\right)$$

(d) The resulting minimum mean-square error for the estimator $\hat{T}_{0,\text{lmmse}}$ is $\left(\sigma_T^{-2} + e^{-2t_1} + e^{-2t_2}\right)^{-1}$. Since $\sigma_T^{-2} > 0$, the LMMSE estimator will have a lower error as compared to BLUE. The estimator $\hat{T}_{0,\text{lmmse}}$ will have a bias $b = 0$ as

$$b = \left(\frac{e^{-2t_1} + e^{-2t_2}}{\sigma_T^{-2} + e^{-2t_1} + e^{-2t_2}} - 1\right)E\{T_0\}.$$

Question 2 (10 points): Linear prediction

Suppose the signal $x(n)$ is *wide-sense stationary*. We develop a first-order linear predictor of the form

$$\hat{x}(n+1) = w(0)x(n) + w(1)x(n-1) = \mathbf{w}^H\mathbf{x}$$

where $\mathbf{w} = [w(0)\ w(1)]^T$ and $\mathbf{x} = [x(n)\ x(n-1)]^T$.

(1pts) (a) Show that the autocorrelation sequence of a wide-sense stationary random process is a conjugate symmetric function of the lag k, i.e., $r_x(k) = r_x^*(-k)$.

(4pts) (b) Derive the optimum $\mathbf{w}$ by minimizing the mean-squared error

$$J(\mathbf{w}) = E\{|\mathbf{w}^H\mathbf{x} - x(n+1)|^2\}.$$

(2pts) (c) Suppose the autocorrelation of $x(n)$ for the first three lags are $r_x(0) = 1$, $r_x(1) = 0$ and $r_x(2) = 1$. To solve the *normal equations* obtained in Part (b) of this question, we will use the *steepest gradient descent algorithm*. Will the steepest-descent algorithm converge if we choose the step size $\mu = 4$, and *why*?

(3pts) (d) To converge to $\mathbf{w}_{\text{opt}}$, how many iterations are required for the steepest-descent algorithm with $r_x(0) = 1$, $r_x(1) = 0$, $r_x(2) = 1$, $\mu = 1$, and $\mathbf{w}^{(0)} = \mathbf{0}$. To answer this question, first compute $\mathbf{w}_{\text{opt}}$.

Solutions

(a) For a wide-sense stationary random process, the auto-correlation function is defined as

$$r_x(k) = E\{x(n+k)x^*(n)\} = E\{x^*(n)x(n+k)\} = r_x^*(-k).$$

(b) The cost function for a first-order linear predictor is

$$J(\mathbf{w}) = E\{|\mathbf{w}^H\mathbf{x} - x(n+1)|^2\} = \mathbf{w}^{\text{H}}\mathbf{R}_x\mathbf{w} - \mathbf{w}^{\text{H}}\mathbf{r}_{xd} - \mathbf{r}_{xd}^{\text{H}}\mathbf{w} + r_x(0)$$

where

$$\mathbf{R}_x = \begin{bmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \end{bmatrix} \quad \text{and} \quad \mathbf{r}_{xd} = E\left\{ \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} x^*(n+1) \right\} = \begin{bmatrix} r_x(1) \\ r_x(2) \end{bmatrix}$$

Then, the optimum solution is

$$\mathbf{w}_{\text{opt}} = \mathbf{R}_x^{-1}\mathbf{r}_{xd}$$

(c) With $r_x(0) = 1$, $r_x(1) = 0$ and $r_x(2) = 1$, $\mathbf{R}_x = \mathbf{I}$ with eigenvalues $\lambda_{\text{min}} = \lambda_{\text{max}} = 1$. For convergence, since we require $0 \le \mu \le \frac{2}{\lambda_{\text{max}}}$, with $\mu = 4$, the steepest gradient descent algorithm will not converge.

(d) The update equations for the steepest gradient descent algorithm is

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \left[\mathbf{w}^{(k)} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right].$$

with $\mathbf{w}^{(1)} = \mathbf{0}$. Therefore, $\mathbf{w}^{(1)} = \mathbf{w}^{(2)} \cdots = \mathbf{w}_{\text{opt}} = [0, 1]^T$.