DYNAMIC TRANSMIT POWER ALLOCATION FOR DISTRIBUTED MIMO RADAR TARGET DETECTION

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ABSTRACT

Distributed multiple-input multiple-output (MIMO) radars can use multiple transmitters and receivers simultaneously to detect targets. In order to maximize the probability of target detection, it is necessary to allocate the available transmit power resources suitably. The optimal allocation requires calculating the probability of detection, which is a computationally complex task, so using the exact distribution is very difficult in a dynamic scenario. In order to alleviate this problem, we propose an approximate distribution that is computationally simpler. This approximation is compared with the exact distribution as well as other cost functions that depend on the distribution of the test statistic, including the Kullback–Leibler divergence. It is demonstrated that the approximate distribution works well in the power allocation problem for MIMO radar target detection.

Index Terms— MIMO radar, Target detection, Power allocation, Optimization

1. INTRODUCTION

Distributed multiple-input multiple-output (MIMO) radar systems use multiple transmitters and receivers in different locations over a large area. Using multiple waveforms from different transmitters and combining multiple measurements, the target detection and parameter estimation capabilities of the radar system can be improved. In order to achieve a good target detection capability within the given power budget, the transmit power for each transmitter needs to be optimized.

Power allocation for distributed radar systems has been studied previously in several papers. Power allocation methods that achieve a predefined MSE for target location estimation have been proposed in [1] and [2]. The power allocation was formulated in [3] as a combinatorial subset selection problem of choosing the transmitters that should be active to reach a desired target localization MSE. Power allocation for target detection was studied in [4], where the exact probability of detection was optimized under probability of false alarm constraints.

Unfortunately, optimizing the exact probability of detection directly is computationally difficult and the optimization problem is non-convex. In a time-varying scenario, the optimal transmit power is constantly varying, so the optimization has to performed in real time. Thus, using the exact probability of detection in the optimization is not plausible in a dynamic scenario with moving transmitters or receivers. It is therefore necessary to find an alternative optimization criterion that still results in a desired probability of detection. To this end, we propose an approximate distribution for the detection test statistic that is far simpler to compute. The approximation is based on the gamma distribution for which the parameters are chosen so that the mean and the variance are equal to those of the true distribution. Furthermore, we compare the proposed approximate distribution with other optimization criteria, namely the Kullback–Leibler divergence, the J-divergence, and the signal to noise ratio (SNR). Numerical examples are provided to show the feasibility of the proposed approximation.

This paper is organized as follows: The signal model is given in Section 2 and target detection is discussed in Section 3. The different optimization criteria are introduced in Section 4, while numerical results are provided in Section 5. Finally, the concluding remarks are given in Section 6.

2. SIGNAL MODEL

In a distributed radar system with M transmitters and N receivers, the baseband received signal at the ith receiver is

$$r_i(t) = \sum_{m=1}^{M} L_{Ri} c_{im} L_{Tm} \phi_{im}(t) a_m s_m(t-\tau_{im}) + \nu_i(t), \quad (1)$$

where L_{Ri} is the path loss to the receiver, c_{im} is the scattering coefficient, L_{Tm} is the path loss from the transmitter, ϕ_{im} is the Doppler and phase factor, a_m is the transmit amplitude, s_m is the waveform, and τ_{im} is the propagation delay, and ν_i is the receiver noise. The noise ν_i is assumed to be zero-mean complex circular Gaussian. Furthermore, Swerling model I is assumed [5], so the scattering coefficients are zero-mean complex circular Gaussian as well. The variables that need to be optimized are the amplitudes a_m for each waveform.

The received signal is filtered before target detection using matched or mismatched filters. The filter output of the ith receiver for the kth waveform can be written as

$$\begin{aligned} y_{ik} =& \sum_t r_i(t)\hat{\phi}^*_{ik}(t)s^*_k(t-\hat{\tau}_{im}) \\ =& \sum_{m,t} L_{Ri}c_{im}L_{Tm}a_m\phi_{im}(t)\hat{\phi}^*_{ik}(t)s_m(t-\tau_{im})s^*_k(t-\hat{\tau}_{im}) \\ +& \sum_t \nu_i(t)\hat{\phi}^*_{ik}(t)s^*_k(t-\hat{\tau}_{im}). \end{aligned} \tag{2}$$

We assume that the transmitted signals have the required auto- and cross-correlation properties so that signals originating from different transmitters can be separated at the receiver filters. As we want to detect a target in a certain range (and Doppler) bin, it is possible align the filtered signals in delay and possibly in Doppler shifts such that

$$y_{ik} = L_{Ri}c_{ik}L_{Tk}a_k + n_i, \tag{3}$$

where n_i is the filtered noise. Stacking the filter outputs into a single vector, one obtains

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} y_{11} & y_{21} & \dots & y_{NM} \end{bmatrix}^T \\ &= (\mathrm{diag}(\mathbf{a}) \otimes \mathbf{I}_N)(\mathbf{I}_M \otimes \mathbf{L}_R)(\mathbf{L}_T \otimes \mathbf{I}_N)\mathbf{c} + \mathbf{n} \\ &= \mathbf{A}(\mathbf{L}_T \otimes \mathbf{L}_R)\mathbf{c} + \mathbf{n}, \end{aligned} \tag{4}$$

where $\mathrm{diag}(\mathbf{x})$ denotes a diagonal matrix with the elements of vector $\mathbf{x}$ on the diagonal, $\mathbf{L}_R = \mathrm{diag}(\begin{bmatrix} L_{R1} & \dots & L_{RN} \end{bmatrix})$, $\mathbf{L}_T = \mathrm{diag}(\begin{bmatrix} L_{T1} & \dots & L_{TM} \end{bmatrix})$, $\mathbf{A} = \mathrm{diag}(\mathbf{a}) \otimes \mathbf{I}_N$, $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \dots & a_M \end{bmatrix}^T$, $\mathbf{c} = \begin{bmatrix} c_{11} & c_{21} & \dots & c_{NM} \end{bmatrix}^T$, $\otimes$ denotes the Kronecker product, $\mathbf{I}_k$ is a $k \times k$ identity matrix, and $\mathbf{n}$ is the stacked filtered noise vector. The $MN \times MN$ covariance matrix of the filter output is

$$\begin{aligned} \mathbf{R}_\mathbf{y} &= \mathrm{E}[\mathbf{y}\mathbf{y}^H] \\ &= \mathbf{A}(\mathbf{L}_T \otimes \mathbf{L}_R)\mathbf{R}_\mathbf{c}(\mathbf{L}_T \otimes \mathbf{L}_R)\mathbf{A} + \mathbf{R}_\mathbf{n} \\ &= \mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n}, \end{aligned} \tag{5}$$

where $\mathbf{R}_\mathbf{c}$ is the covariance matrix of the scattering amplitudes, $\mathbf{R}_\mathbf{s}$ is the covariance matrix of the signal (which is a function of the amplitudes $\mathbf{a}$ as well as the path losses $\mathbf{L}_T$ and $\mathbf{L}_R$), and $\mathbf{R}_\mathbf{n}$ is the covariance matrix of the filtered noise. The receiver noise is assumed to be Gaussian with the covariance matrix $\mathbf{R}_{\boldsymbol{\nu}}$, so the filtered noise has the covariance matrix

$$\mathbf{R}_\mathbf{n} = \boldsymbol{\Xi} \otimes \mathbf{R}_{\boldsymbol{\nu}}, \tag{6}$$

where $\boldsymbol{\Xi}$ is the matrix ambiguity function of the transmitted waveforms. It is assumed that the transmitted waveforms are designed so that $\boldsymbol{\Xi}$ is invertible. Moreover, $\mathbf{R}_\mathbf{n}$ is assumed to be positive definite due to thermal noise.

3. TARGET DETECTION

Target detection is most commonly based on binary hypothesis testing. Given the target location $\mathbf{x}$ and the target velocity $\mathbf{v}$, the null hypothesis $\mathcal{H}_0$ is that no target is at $(\mathbf{x}, \mathbf{v})$, whereas the alternative hypothesis $\mathcal{H}_1$ is that a target is present at $(\mathbf{x}, \mathbf{v})$. The likelihood ratio test (LRT) is known to be an optimal testing procedure in the Neyman–Pearson sense for the binary hypothesis test. Since the scattering amplitude vector $\mathbf{c}$ and filtered noise vector $\mathbf{n}$ are assumed to be complex circular Gaussian, the matched filter output vector is distributed as

$$\mathbf{y} \sim \begin{cases} \mathcal{CN}(0, \mathbf{R}_\mathbf{n}), & \mathcal{H}_0 \\ \mathcal{CN}(0, \mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n}), & \mathcal{H}_1 \end{cases} \tag{7}$$

The log-likelihood ratio is thus

$$\begin{aligned} \mathcal{L} = & \mathbf{y}^H[\mathbf{R}_\mathbf{n}^{-1} - (\mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n})^{-1}]\mathbf{y} \\ & + \det[(\mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n})^{-1}] - \det(\mathbf{R}_\mathbf{n}^{-1}). \end{aligned} \tag{8}$$

The test statistic can be identified to be $\mathbf{y}^H[\mathbf{R}_\mathbf{n}^{-1} - (\mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n})^{-1}]\mathbf{y}$. Assuming $\mathbf{z}$ is a standard complex normal with a zero mean and an identity covariance matrix, the distribution of the test statistic under the null hypothesis can be written as

$$\begin{aligned} & \mathbf{y}^H[\mathbf{R}_\mathbf{n}^{-1} - (\mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n})^{-1}]\mathbf{y} \\ & \sim \mathbf{z}^H\mathbf{R}_\mathbf{n}^{1/2}[\mathbf{R}_\mathbf{n}^{-1} - (\mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n})^{-1}]\mathbf{R}_\mathbf{n}^{1/2}\mathbf{z} \\ & \sim \mathbf{z}^H[\mathbf{I}_{MN} - (\mathbf{S} + \mathbf{I}_{MN})^{-1}]\mathbf{z}, \end{aligned} \tag{9}$$

where the SNR matrix is defined as $\mathbf{S} = \mathbf{R}_\mathbf{s}\mathbf{R}_\mathbf{n}^{-1}$. The noise covariance matrix $\mathbf{R}_\mathbf{n}$ can be assumed to be invertible, see (6). $\mathbf{S}$ is positive semidefinite (it has the same eigenvalues as $\mathbf{R}_\mathbf{n}^{-1/2}\mathbf{R}_\mathbf{s}\mathbf{R}_\mathbf{n}^{-1/2}$), and thus, $\mathbf{I}_{MN} - (\mathbf{S} + \mathbf{I}_{MN})^{-1}$ is positive semidefinite as well.

Under the alternative hypothesis, the distribution is

$$\begin{aligned} & \mathbf{y}^H[\mathbf{R}_\mathbf{n}^{-1} - (\mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n})^{-1}]\mathbf{y} \\ & \sim \mathbf{z}^H(\mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n})[\mathbf{R}_\mathbf{n}^{-1} - (\mathbf{R}_\mathbf{s} + \mathbf{R}_\mathbf{n})^{-1}]\mathbf{z} \\ & \sim \mathbf{z}^H\mathbf{S}\mathbf{z} \end{aligned} \tag{10}$$

The probability distribution function of the test statistic can be given using the eigenvalues of $\mathbf{S}$ and the series formula of gamma distribution functions given in [6], but this is numerically complex to calculate.

It should be noted that the eigenvalues of the SNR matrix $\mathbf{S}$ are invariant with respect to linear full-rank transformations of the filter output $\mathbf{y}$, and whitening or other such operations cannot be used to improve the probability of detection.

4. POWER ALLOCATION

The goal of the MIMO radar system is to detect the targets in a predefined surveillance area. This surveillance area is divided into range and Doppler bins for detection and the set of bins is denoted by $\mathcal{X}$. With the assumption that the target velocity does not affect the signal power, the Doppler may be disregarded in the optimization. For a given power budget of P_{tot}, we would like to allocate the power so that the probability of detection is as high as possible over the set $\mathcal{X}$ with a constraint on the probability of false alarm p_f. Equivalently, we can minimize the maximum probability of missed detection p_m over the surveillance area. The problem may be formulated as

$$\min_{\mathbf{a}} \max_{\mathbf{x}\in\mathcal{X}} p_m(\mathbf{a},\mathbf{x}) \text{ s.t. } \mathbf{a}^T\mathbf{a} \le P_{\text{tot}},\ p_f \le p_{f,\min}, \tag{11}$$

where $\mathbf{x}$ is the target position (range bin). Each point $\mathbf{x}$ has an SNR matrix associated with it that depends on the transmit amplitudes $\mathbf{a}$ as well as the path losses $\mathbf{L}_T$ and $\mathbf{L}_R$.

Based on (3), we can assume that the target velocity does not affect the probability of detection, as the velocity changes the phase of the received signal, not the power. Furthermore, we have assumed a total power constraint $\mathbf{a}^T\mathbf{a} \le P_{\text{tot}}$. Other constraints on the amplitudes are possible as well, including a maximum power constraint for each transmitter. The expression for the probability of missed detection p_m can be found in [4].

Using the gamma series formula given in [6] is computationally complex as thousands or even tens of thousands of terms might be needed for accurate approximation, in particular when the condition number of the SNR matrix is large. If all the eigenvalues of the SNR matrix are distinct, a closed-form expression for the distribution function of the test statistic can be computed (see e.g. [7]). However, this approach is numerically unstable. We therefore need to find an alternative criterion that provides an accurate approximation for the exact probability of missed detection so that the transmit power allocation can be done in a dynamic scenario.

In order to facilitate the detection, we want to make the distribution of the test statistic as different as possible under the null hypothesis and the alternative hypothesis. A common measure for the similarity of two distributions is the Kullback–Leibler divergence also known as the relative entropy [8]. For complex Gaussian distributions with covariance matrices $\mathbf{R}_0$ and $\mathbf{R}_1$ and equal mean, the Kullback–Leibler divergence is given by

$$\begin{aligned}\mathcal{D}_{\text{KL}}(\mathbf{R}_1|\mathbf{R}_0) &= \operatorname{tr}(\mathbf{R}_0^{-1}\mathbf{R}_1) - \operatorname{tr}(\mathbf{I}) \\ &\quad + \log\det(\mathbf{R}_0) - \log\det(\mathbf{R}_1).\end{aligned} \tag{12}$$

The Kullback–Leibler divergence is not concave, so maximizing it is not a convex problem. Substituting $\mathbf{R}_0 = \mathbf{R_n}$ and $\mathbf{R}_1 = \mathbf{R_s} + \mathbf{R_n}$, one obtains

$$\mathcal{D}_{\text{KL}}(\mathbf{R}_1|\mathbf{R}_0) = \operatorname{tr}(\mathbf{S}) - \log\det(\mathbf{S} + \mathbf{I}_{MN}). \tag{13}$$

A metric that is related to the Kullback–Leibler divergence but symmetric is the J-divergence, defined as [9]

$$\begin{aligned}\mathcal{D}_{\text{J}}(\mathbf{R}_1,\mathbf{R}_0) &= \mathcal{D}_{\text{KL}}(\mathbf{R}_1|\mathbf{R}_0) + \mathcal{D}_{\text{KL}}(\mathbf{R}_0|\mathbf{R}_1) \\ &= \operatorname{tr}(\mathbf{S} + \mathbf{I}_{MN}) + \operatorname{tr}[(\mathbf{S} + \mathbf{I}_{MN})^{-1}].\end{aligned} \tag{14}$$

The J-divergence is a convex function, so maximizing it is not a convex problem.

As a simplification of the Kullback–Leibler divergence, we can drop the determinant and only consider the trace of the SNR matrix. It is straightforward to show that the trace is quadratic in the amplitudes $\mathbf{a}$, so the minimax problem is yet again non-convex. Only in the special case in which the $\mathbf{R_n}$ is diagonal, the trace is a linear function of the transmit amplitudes squared and the maximization is a convex problem.

Denoting the eigenvalues of the SNR matrix by λ_i, the Kullback–Leibler divergence and the J-divergence can be expressed as

$$\mathcal{D}_{\text{KL}} = \sum_i \lambda_i + \log(\lambda_i + 1) \tag{15}$$

$$\mathcal{D}_{\text{J}} = \sum_i \lambda_i + 1 + \frac{1}{\lambda_i + 1}, \tag{16}$$

respectively. We can immediately see that both these divergences behave like the trace of the SNR matrix for large eigenvalues. When optimizing the probability of detection, we want the SNR matrix to have as large eigenvalues as possible. Thus, the Kullback–Leibler divergence, the J-divergence, and the trace are likely to yield a similar power allocation.

We propose another optimization criterion that is based on an approximation of the distribution of the test statistic. The exact distribution can be written as a series of gamma distribution functions, but we use a single gamma distribution that has equal mean and variance with the test statistic. The rationale for this type of approximation is discussed in [10].

As seen in the previous section, the test statistic can be written as $\mathbf{z}^H\mathbf{X}\mathbf{z}$, where $\mathbf{z}$ is a standard complex Gaussian random vector and $\mathbf{X}$ is a positive-semidefinite matrix. The mean and the variance of $\mathbf{z}^H\mathbf{X}\mathbf{z}$ are the given by $\operatorname{tr}(\mathbf{X})$ and $\operatorname{tr}(\mathbf{X}^2)$, respectively. A standard gamma distribution with parameters α and β has a mean $\alpha\beta$ and a variance $\alpha\beta^2$. In order to have these equal to those of the test statistic, one needs

$$\alpha = \operatorname{tr}^2(\mathbf{X})/\operatorname{tr}(\mathbf{X}^2) \tag{17}$$

$$\beta = \operatorname{tr}(\mathbf{X}^2)/\operatorname{tr}(\mathbf{X}), \tag{18}$$

where $\mathbf{X}$ would be $[\mathbf{I}_{MN} - (\mathbf{S} + \mathbf{I}_{MN})^{-1}]$ under the null hypothesis and $\mathbf{S}$ under the alternative hypothesis, where $\mathbf{S}$ depends on the transmit amplitudes $\mathbf{a}$ that are to be optimized. One can then use this distribution to form an approximate probability of detection and use that in the optimization.

Although the exact distribution of the test statistic can be written as a series of gamma distribution functions, accurate approximation of this requires typically hundreds of evaluations of the gamma distribution function. Given that the proposed approximation requires only a single evaluation, the computational complexity is significantly reduced.

5. EXAMPLES

In this section, we compare the different optimization criteria for the transmit power optimization. The probability of de-

Table 1: Transmit power allocation for maximizing the minimum probability of detection p_d. The trace of the SNR matrix, Kullback–Leibler divergence, and the J-divergence result in the same allocation that is far from the optimal. Using the proposed approximate results in a significantly better probability of detecting the target.

Method	Tx Power			p_d
SNR matrix trace	0.0000	0.0000	3.0000	0.917
Approx. p_m	0.0000	1.5362	1.4638	0.9854
True p_m	0.8694	1.2168	0.9139	0.9988
J-div.	0.0000	0.0000	3.0000	0.917
KL-div	0.0000	0.0000	3.0000	0.917

tection is optimized over the 2-D rectangle $[0, 1] \times [0, 1]$, in arbitrary units. There are three transmitters, located at (0.3 0.3), (0.5 0.2), and (0.8 0.25) with total available power equal to three. Four receivers are located at (0.15, 0.6), (0.38, 0.26), (0.60, 0.14), and (0.95, 0.55). In addition to the thermal noise, there is a narrowband jammer (0.8, 2.0) with a jammer to noise ratio equal to 3dB. The scattering amplitudes $\mathbf{c}$ are independent with unit variance. The path losses coefficients $\mathbf{L}_T$ and $\mathbf{L}_R$ at location $\mathbf{x}$ were simply assumed to be $\|\mathbf{x}-\mathbf{x}_m\|^{-2}$ and $\|\mathbf{x} - \mathbf{x}_n\|^{-2}$, where $\mathbf{x}_m$ is the location of the transmitter and $\mathbf{x}_n$ that of the receiver. The probability of false alarm was constrained to 10^{-3} in the Neyman–Pearson detector. The surveillance area was divided into (range) bins and the optimization criteria were calculated for each bin. The transmit amplitudes were then optimized using Matlab `fmincon` function starting from a uniform power allocation minimizing the $\max p_m$ over all the bins.

The results for the transmit power optimization are shown in Table 1. Maximizing the minimum trace of the SNR matrix, the KL-divergence, or the J-divergence all result in an allocation where all the transmitted power is transmitted from the rightmost transmitter, which leads to a minimum probability of detection of 0.917 in the surveillance area. This corroborates that these criteria are closely related .

The proposed distribution allocates the power almost evenly between the second and the third transmitter, resulting in a minimum p_d equal to 0.9854. Optimization using the actual distribution yields a power allocation of 0.8694, 1.2168, 0.9139 with a probability of missed detection of 0.9988.

Fig.1 shows the optimized trace of the SNR matrix, the Kullback–Leibler divergence, and the J-divergence on the surveillance area. As expected, these criteria are very similar. The approximate and the exact probability of missed detection are compared in Fig.2. The approximation is conservative in the sense that it produces slightly larger probability of missed detection compared to the exact distribution.

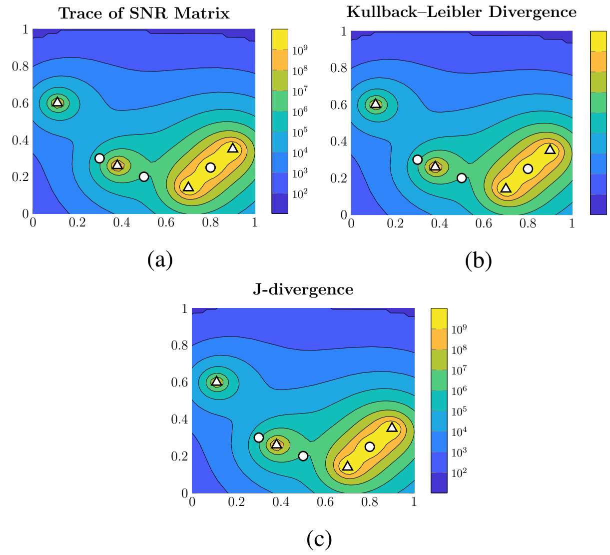

Fig. 1: Comparison of (a) the trace of the SNR matrix, (b) the Kullback–Leibler divergence, and (c) J-divergence. The transmitters are denoted by circles and the receivers by triangles. These three criteria are all very similar.

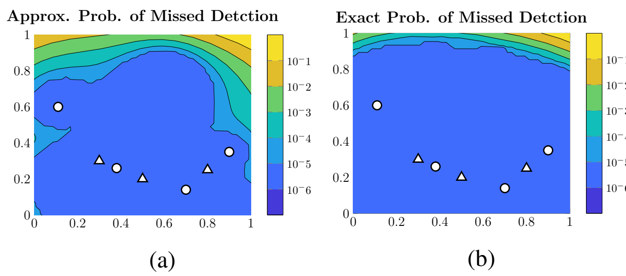

Fig. 2: Comparison of (a) approximate and (b) the exact probability of missed detection. The transmitters are denoted by circles and the receivers by triangles. The approximation produces slightly larger probability of missed detection.

6. CONCLUSIONS

Transmit power optimization for distributed MIMO radar target detection has been discussed in this paper. For a dynamic scenarios, it is necessary to have a low-complexity method so that the transmit power allocation can be done in real-time. Due to the computational complexity and numerical stability problems, the exact distribution of the test statistic for target detection is not suitable for this task. We proposed an alternative approximate distribution that has significantly lower computational complexity. This distribution was compared to optimizing divergence measures that all produced similar but sub-optimal result. Compared to these, the proposed approximate distribution provided a much better probability of detection in the surveillance area.

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