Sparse Sensing for Statistical Inference

Model-driven and data-driven paradigms

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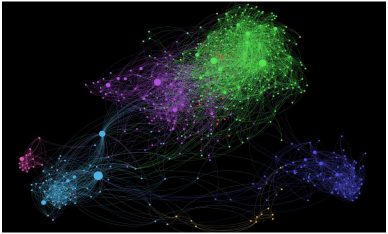
Power networks, grid analytics

Health informatics

Radio astronomy (e.g., SKA)

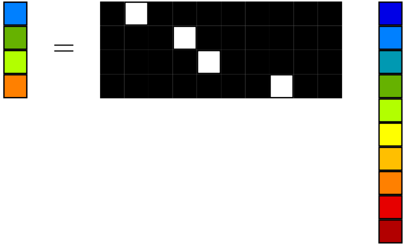


Internet, social media

Design sparse sensing functions

What is sparse sensing?

$\mathbf{y} \in \mathbb{R}^{d}$ $\quad \boldsymbol{\Phi}(\mathbf{w}) = \overbrace{\mathrm{diag_r}(\mathbf{w})}^{\{0,1\}}$ $\quad \mathbf{x} \in \mathbb{R}^{D}$



What is sparse sensing?

Design $\mathbf{w} \in \{0,1\}^D$ to select the most "informative" d ($\ll D$) samples for

- data acquisition (e.g., offline sensor selection)
- data selection (e.g., outlier rejection, censoring)
- Or, combination of above —hybrid

State of the art

- Compressive sensing: sparse signal recovery

  [Donoho-2006], [Candès-Wakin-2008]

- Sensor selection: model-driven

  - convex optimization: design $\{0,1\}^M$ selection vector

    [Joshi-Boyd-2009], [Chepuri-Leus-2015]

  - greedy methods and heuristics: submodularity

    [Krause-Singh-Guestrin-08], [Ranieri-Chebira-Vetterli-14]

- Censoring (or data selection) and outlier rejection: data-driven

  [Rago-Willett-Shalom-96], [Msechu-Giannakis-12]

Model-driven, data-driven, or hybrid?

Linear regression setup— model-driven

- Observations follow

$$x_m = \mathbf{a}_m^T \boldsymbol{\theta} + n_m,\ m = 1, 2, \ldots, D$$

  - $\boldsymbol{\theta} \in \mathbb{R}^p$ unknown parameter
  - n_m i.i.d. zero-mean unit-variance Gaussian noise

Problem statement

Given $\{\mathbf{a}_m\}$ and noise pdf, i.e., only the data model, design $\mathbf{w}$ to select the best subset of d ($\ll D$) sensors

- Best subset of d ($\ll D$) sensors are obtained by optimizing a scalar function (trace, max. eigenvalue, log det) of the CRB matrix of $\boldsymbol{\theta}$:

$$f(\mathbf{w}) = g\left\{\left(\sum_{m=1}^{D} w_m \mathbf{a}_m \mathbf{a}_m^{\mathsf{T}}\right)^{-1}\right\}$$

Optimization problem

- Sparse sensing function can be designed by solving

$$\min_{\mathbf{w}\in\mathcal{W}} \; f(\mathbf{w})$$

$\mathcal{W} = \{\mathbf{w} \in \{0,1\}^D \mid \|\mathbf{w}\|_0 = d\}.$

- The above problem is convex on $\mathbf{w}$ if the set $\mathcal{W}$ is relaxed to $\mathcal{W}_{\mathrm{c}} = \{\mathbf{w} \in [0,1]^D \mid \|\mathbf{w}\|_1 = d\}.$
- Monotone submodular cost functions can be optimized using near-optimal greedy methods.

\+ Sampler needs to be designed offline only once, and samplers are obtained by optimizing ensemble performance.

\- Design might suffer from any possible outliers or model mismatch.

Model-driven design for other inference tasks

- Can be generalized to nonlinear models by optimizing scalar functions of the Cramér-Rao bound matrix.

  S.P. Chepuri and G. Leus. Sparsity-Promoting Sensor Selection for Non-linear Measurement Models. *IEEE Trans. on Signal Processing, Volume 63, Issue 3, pp. 684-698, February 2015*.

- Samplers for nonlinear models with correlated errors can be obtained by solving a convex program.

  S.P. Chepuri and G. Leus. Sparse Sensing for Estimation with Correlated Observations. In Proc. of Asilomar 2015, Pacific Grove, CA, Nov. 2015.

- Samplers can be designed for detection problems by optimizing Kullback-Leibler or Bhattacharyya distance measures.

  S.P. Chepuri and G. Leus. *Sparse Sensing for Distributed Detection*. *Trans. on Signal Processing*, October 2015.

Linear regression setup— data-driven

- The output data $\{x_m\}_{m=1}^{D}$ is possibly contaminated with up to o outliers.
- We know the (uncontaminated) data model

$$\bar{x}_m = \mathbf{a}_m^T \boldsymbol{\theta} + n_m,\ m = 1, 2, \ldots, D$$

- $\boldsymbol{\theta} \in \mathbb{R}^p$ unknown parameter
- n_m i.i.d. zero-mean unit-variance Gaussian noise

Problem statement

Given $\{x_m\}$, $\{\mathbf{a}_m\}$, and noise pdf:

(a) design $\mathbf{w}$ to censor less-informative samples and reject outliers

(b) estimate $\boldsymbol{\theta}$ that performs using the uncensored data

Data-driven design

- Data samples with smaller residuals are informative; more generally the one's with large likelihood
- Sensing function is obtained by solving

$$\min_{\mathbf{w}\in\mathcal{W},\boldsymbol{\theta}} \sum_{m=1}^{D} w_m \left(x_m - \mathbf{a}_m^T\boldsymbol{\theta}\right)^2 \Leftrightarrow \min_{\mathbf{w}\,\in\,\mathcal{W}} r(\mathbf{w})$$

$$r(\mathbf{w}) = \mathbf{x}_w^\mathsf{T}\left(\mathbf{I} - \mathbf{A}_w\left(\mathbf{A}_w^\mathsf{T}\mathbf{A}_w\right)^{-1}\mathbf{A}_w^\mathsf{T}\right)\mathbf{x}_w.$$
$$\mathbf{A}_w = \mathrm{diag}_r(\mathbf{w})\mathbf{A};\, \mathbf{x}_w = \mathrm{diag}_r(\mathbf{w})\mathbf{x}.$$

- Also known as least trimmed squares.
- This problem is non-convex in general
  - Can be convexified for linear Gaussian case
  - Markov chain Monte Carlo methods (e.g., Metropolis-Hastings sampling)

Generalization of data-driven design

- For linear Gaussian models, the cost function amounts to sparsity based outlier rejection methods

  [Fuchs-1999], [Rousseeuw-Leroy-2005], [Giannakis et al.-2011]

- Can be generalized to nonlinear models by optimizing likelihood function parameterized with $\boldsymbol{\theta}$ and $\mathbf{w}$.

  G. Kail, S.P. Chepuri, and G. Leus. Robust Censoring Using Metropolis-Hastings Sampling. *IEEE Journ. of Selec. Topics in Signal Processing*, Nov. 2015.

+ Designs are robust to outliers.

- Sampler needs to be designed for each data realization, and samplers are obtained by optimizing instantaneous measure (performance could be bad).

Hybrid Model-data-driven design

- Data-driven samplers are robust to outliers, but don't take the resulting inference performance (i.e., MSE) into account

- Model-driven samplers are MSE optimal, but are not robust to outliers.

- Hybrid model-data-driven designs allow to combine (and trade) the above two advantages.

+ Designs are robust to outliers, and take into account the inference performance.

- Sampler needs to be designed for each data realization.

Optimization problem

- Hybrid model-data-driven sensing scheme jointly optimizes the likelihood function (i.e., residual) and the MSE:

$$\min_{\mathbf{w} \in \mathcal{W}} \; r(\mathbf{w}) + \lambda f(\mathbf{w})$$

$\mathcal{W} = \{\mathbf{w} \in \{0,1\}^D \mid \|\mathbf{w}\|_0 = d\}$.

- $\lambda \to 0(\infty)$ results in the related data (model)-driven scheme

- This problem is non-convex in general
  - Can be convexified for linear Gaussian case
  - Markov chain Monte Carlo methods (e.g., Metropolis-Hastings sampling)

Convex optimization based solver

- Hybrid model-data-driven sensing design is equivalent to

$$
\begin{aligned}
\min_{\mathbf{w} \in \mathcal{W}, t_1, t_2} \quad & t_1 + \lambda t_2 \\
\text{s.t.} \quad & r(\mathbf{w}) \leq t_1, \\
& f(\mathbf{w}) \leq t_2.
\end{aligned}
$$

- Using Schur complement and $\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi} = \mathrm{diag}(\mathbf{w})$

$$
\begin{aligned}
\min_{\mathbf{w} \in \mathcal{W}, t_1, t_2} \quad & t_1 + \lambda t_2 \\
\text{s.t.} \quad & \begin{bmatrix} \mathbf{A}^{\mathsf{T}}\mathrm{diag}(\mathbf{w})\mathbf{A} & \mathbf{A}^{\mathsf{T}}\mathrm{diag}(\mathbf{w})\mathbf{x} \\ \mathbf{x}^{\mathsf{T}}\mathrm{diag}(\mathbf{w})\mathbf{A} & t_1 - \mathbf{x}^{\mathsf{T}}\mathrm{diag}(\mathbf{w})\mathbf{x} \end{bmatrix} \succeq \mathbf{0}, \\
& f(\mathbf{w}) \leq t_2.
\end{aligned}
$$

- The above problem is convex after relaxing $\mathcal{W}$ to $\mathcal{W}_c$.

Metropolis-Hastings sampler

- Powerful concept for generating samples from complex distributions:
  1. Each iteration generates a proposal $\widetilde{\mathbf{w}}$ from some proposal distribution $q\big(\widetilde{\mathbf{w}}\big|\mathbf{w}^{(j-1)}\big)$.
  2. The new sample $\mathbf{w}^{(j)}$ is obtained as

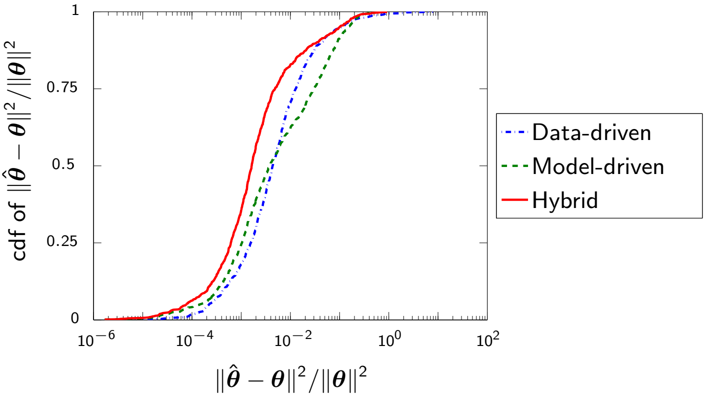
$$\mathbf{w}^{(j)} = \begin{cases} \widetilde{\mathbf{w}} & \text{with probability } \alpha_j \\ \mathbf{w}^{(j-1)} & \text{with probability } 1-\alpha_j \end{cases}$$

$$\alpha_j = \min\left\{ \frac{p_{\mathrm{t}}\big(\widetilde{\mathbf{w}}\big)\, q\big(\mathbf{w}^{(j-1)}\big|\widetilde{\mathbf{w}}\big)}{p_{\mathrm{t}}\big(\mathbf{w}^{(j-1)}\big)\, q\big(\widetilde{\mathbf{w}}\big|\mathbf{w}^{(j-1)}\big)}, 1 \right\}$$

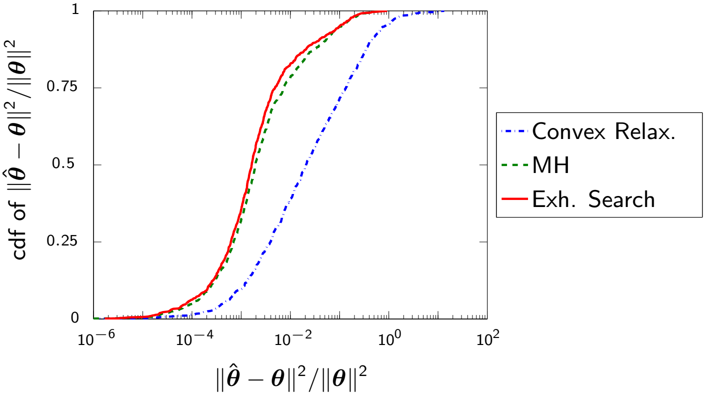
$$p_{\mathrm{t}}(\mathbf{w}) \propto \exp\left(-\frac{1}{2\,\sigma^2}\left(r(\mathbf{w}) + \lambda f(\mathbf{w}) \right) \right)$$

- Doesn't involve convex relaxations ($\mathcal{W}$ to $\mathcal{W}_c$).

Numerical experiments



- Best performance of the hybrid scheme is not necessarily in between the best performances of the data-driven and model-driven schemes.



- Error distribution achieved with the MH method almost coincides with that of exhaustive search.

Conclusions and future directions

- Model-driven, data-driven, hybrid sparse sensing
  - for basic inference problems
  - respective strengths and weaknesses

- Future directions
  - Correlated observations, clustering, and classification
  - Greedy algorithms (submodular)

Thank You!!