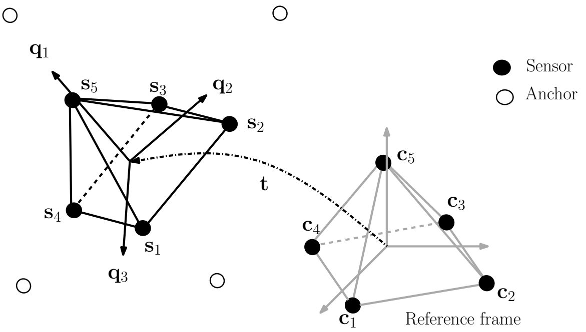
Rigid Body Localization and Tracking

Can we jointly estimate position and orientation using range-only measurements?

Wireless sensors on a rigid body



Rigid body transformation:

$$\mathbf{s}_n = \mathbf{Q}\mathbf{c}_n + \mathbf{t} \quad \Rightarrow \quad \mathbf{S} = \mathbf{Q}\mathbf{C} + \mathbf{t}\mathbf{1}_N^T = \overbrace{\left[\begin{array}{c|c} \mathbf{Q} & \mathbf{t} \end{array}\right]}^{\boldsymbol{\Theta}} \overbrace{\left[\begin{array}{c} \mathbf{C} \\ \hline \mathbf{1}_N^T \end{array}\right]}^{\mathbf{C}_e}$$

rotation matrix $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \in \mathcal{V}_{3,3}$

translation vector $\mathbf{t} \in \mathbb{R}^{3\times 1}$

"Interesting optimization problems on the Stiefel manifold"

Measurement model

Squared-range between the m-th anchor and n-th sensor

$$
\begin{aligned}
d_{mn} &= \|\mathbf{a}_m - \mathbf{s}_n\|_2^2 + n_{mn} \\
&= \|\mathbf{a}_m\|^2 - 2\mathbf{a}_m^T\mathbf{s}_n + \|\mathbf{s}_n\|^2 + n_{mn}
\end{aligned}
$$

$\mathbf{s}_n$: coordinates of the nth sensor
$\mathbf{a}_m$: coordinates of the mth anchor

Collect measurements from M anchors

$$
\mathbf{d}_n = \mathbf{a} - 2\mathbf{A}^T\mathbf{s}_n + \|\mathbf{s}_n\|^2\mathbf{1}_M + \mathbf{n}_n
$$

$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M] \in \mathbb{R}^{3\times M}$
$\mathbf{a} = [\|\mathbf{a}_1\|^2, \|\mathbf{a}_2\|^2, \ldots, \|\mathbf{a}_M\|^2]^T$

Measurement model

$$\mathbf{d}_n = \mathbf{a} - 2\mathbf{A}^T\mathbf{s}_n + \|\mathbf{s}_n\|^2\mathbf{1}_M + \mathbf{n}_n$$

- Eliminate $\|\mathbf{s}_n\|^2\mathbf{1}_M$
- Whiten $\mathbf{R} = \mathbb{E}\{\mathbf{n}_n\mathbf{n}_n^T\} = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_M^2) \in \mathbb{R}^{M\times M}$

$$\mathbf{W}\mathbf{U}_M^T(\mathbf{d}_n - \mathbf{a}) = -2\mathbf{W}\mathbf{U}_M^T\mathbf{A}^T\mathbf{s}_n + \mathbf{W}\mathbf{U}_M^T\mathbf{n}_n,$$

$$\mathbf{P}_M = \mathbf{I}_M - \frac{1}{M}\mathbf{1}_M\mathbf{1}_M^T = \mathbf{U}_M\mathbf{U}_M^T \ \in \mathbb{R}^{M\times M}$$

$$\mathbf{W}(\mathbf{U}_M^T\mathbf{R}\mathbf{U}_M)\mathbf{W} = \mathbf{I}_{M-1}$$

Stacking for all the N sensors

$$\mathbf{W}\mathbf{U}_M^T\mathbf{D} = -2\mathbf{W}\mathbf{U}_M^T\mathbf{A}^T\mathbf{S} + \mathbf{W}\mathbf{U}_M^T\mathbf{N}$$

$$\mathbf{D} = [\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_N] - \mathbf{a}\mathbf{1}_N^T \in \mathbb{R}^{M\times N}$$

Measurement model

$$\mathbf{W}\mathbf{U}_M^T\mathbf{D} = -2\mathbf{W}\mathbf{U}_M^T\mathbf{A}^T\mathbf{S} + \mathbf{W}\mathbf{U}_M^T\mathbf{N}$$

writing compactly

$$\overbrace{\bar{\mathbf{D}}}^{(M-1)\times N} = \overbrace{\bar{\mathbf{A}}}^{(M-1)\times 3}\mathbf{S} + \bar{\mathbf{N}},$$

where $\bar{\mathbf{N}}$ is row-wise white.

Substitute "$\mathbf{S} = \mathbf{\Theta}\mathbf{C}_e$"

$$\boxed{\bar{\mathbf{D}} = \bar{\mathbf{A}}\mathbf{\Theta}\mathbf{C}_e + \bar{\mathbf{N}}}$$

or vectorize it

$$\boxed{\bar{\mathbf{d}} = (\mathbf{C}_e^T \otimes \bar{\mathbf{A}})\boldsymbol{\theta} + \bar{\mathbf{n}}}$$

$$\boldsymbol{\theta} = [\mathbf{q}_1^T, \mathbf{q}_2^T, \mathbf{q}_3^T, \mathbf{t}^T]^T \in \mathbb{R}^{12\times 1}$$

Proposed estimators: static case

Unconstrained (unweighted) Least-Squares

$$\hat{\boldsymbol{\theta}}_{\mathrm{LS}} = \arg\min_{\boldsymbol{\theta}} \quad \|\bar{\mathbf{d}} - (\mathbf{C}_{\mathrm{e}}^{T} \otimes \bar{\mathbf{A}})\boldsymbol{\theta}\|_2^2$$
$$= (\mathbf{C}_{\mathrm{e}}^{T} \otimes \bar{\mathbf{A}})^{\dagger}\bar{\mathbf{d}}.$$

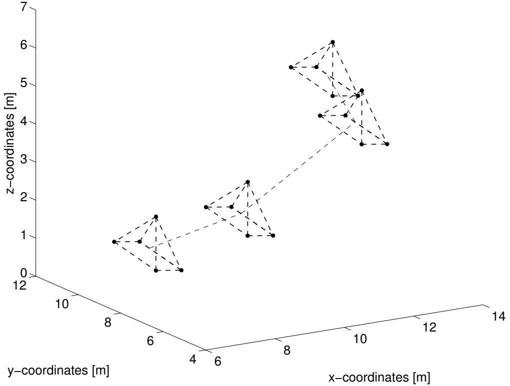
$$\hat{\boldsymbol{\Theta}}_{\mathrm{LS}} = \mathrm{vec}^{-1}(\hat{\boldsymbol{\theta}}_{\mathrm{LS}}) = \left[\ \hat{\mathbf{Q}}_{\mathrm{LS}} \mid \hat{\mathbf{t}}_{\mathrm{LS}}\ \right].$$

Project the LS solution onto the constraint set

$$\hat{\mathbf{Q}}_{\mathrm{UC}} = \arg\min_{\mathbf{Q}} \|\mathbf{Q} - \hat{\mathbf{Q}}_{\mathrm{LS}}\|_F^2 \quad \mathrm{s.to} \quad \mathbf{Q} \in \mathcal{V}_{3,3}$$
$$= (\hat{\mathbf{Q}}_{\mathrm{LS}}\hat{\mathbf{Q}}_{\mathrm{LS}}^{T})^{-1/2}\hat{\mathbf{Q}}_{\mathrm{LS}}.$$

This is a special case of the orthogonal Procrustes problem.
(*Hint:* $\hat{\mathbf{Q}}_{\mathrm{LS}} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{T}$).

Tracking a mobile rigid body



Rigid body kinematics is determined by

1. linear velocity: rate of change of translation
2. angular velocity: rate of change of rotation

State-space model: dynamics

Translational displacement (constant velocity model)

$$\boxed{\mathbf{t}_k = \mathbf{t}_{k-1} + \tau_s \dot{\mathbf{t}}_k + \mathbf{z}_{t,k}}$$

τ_s is the sampling time
$\dot{\mathbf{t}}_k$ is the linear velocity
$\mathbb{E}\{\mathbf{z}_{t,k}\mathbf{z}_{t,k}^T\} = \sigma_t^2 \mathbf{I}_3$

Rotation update equation:

$$\mathbf{Q}_k = \mathbf{F}_{Q,k}\mathbf{Q}_{k-1} + \mathbf{Z}_{Q,k}, \quad \mathbf{F}_{Q,k} \in \mathcal{V}_{3,3}$$

$\mathbf{F}_{Q,k}$ is a function of angular velocity

Vectorizing

$$\boxed{\mathbf{q}_k = (\mathbf{I}_3 \otimes \mathbf{F}_{Q,k})\mathbf{q}_{k-1} + \mathbf{z}_{q,k}}$$

$\mathbf{z}_{q,k} = \mathrm{vec}(\mathbf{Z}_{Q,k})$
$\mathbb{E}\{\mathbf{z}_{q,k}\mathbf{z}_{q,k}^T\} = \sigma_q^2 \mathbf{I}_9.$

State-space model

Stacking translation and rotation update equations:

$$\boldsymbol{\theta}_k = \mathbf{F}_k \boldsymbol{\theta}_{k-1} + \mathbf{u}_k + \mathbf{z}_k \qquad \textit{state equations}$$

$$\bar{\mathbf{d}}_k = \mathbf{H}\boldsymbol{\theta}_k + \bar{\mathbf{n}}_k \qquad \textit{measurement equations}$$

Transition matrix: $\mathbf{F}_k = \mathrm{diag}(\mathbf{I}_3 \otimes \mathbf{F}_{Q,k}, \mathbf{I}_3) \in \mathbb{R}^{12\times 12}$
Control vector: $\mathbf{u}_k = \tau_s[\mathbf{0}_9^T, \dot{\mathbf{t}}_k^T]^T \in \mathbb{R}^{12\times 1}$
Process noise: $\mathbf{z}_k = [\mathbf{z}_{q,k}^T, \mathbf{z}_{t,k}^T]^T \in \mathbb{R}^{12\times 1}$
Covariance matrix: $\mathbf{M} = \mathbb{E}\{\mathbf{z}_k \mathbf{z}_k^T\} = \mathrm{diag}(\sigma_Q^2 \mathbf{I}_9, \sigma_t^2 \mathbf{I}_3)$

Kalman filter: weighted LS formulation

Prediction step:

$$\hat{\boldsymbol{\theta}}_{k|k-1} = \mathbf{F}_k \hat{\boldsymbol{\theta}}_{k-1} + \mathbf{u}_k$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{M}$$

$$\mathbb{E}\{(\hat{\boldsymbol{\theta}}_{k|k-1} - \boldsymbol{\theta}_k)(\hat{\boldsymbol{\theta}}_{k|k-1} - \boldsymbol{\theta}_k)^T = \mathbf{P}_{k|k-1}$$

$$\mathbb{E}\{(\hat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}_{k-1})(\hat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}_{k-1})^T\} = \mathbf{P}_{k-1}$$

"View" $\hat{\boldsymbol{\theta}}_{k|k-1}$ as a noisy measurement,

$$\hat{\boldsymbol{\theta}}_{k|k-1} = \boldsymbol{\theta}_k + \mathbf{e}_{k|k-1}$$

$$\mathbb{E}\{\mathbf{e}_{k|k-1} \mathbf{e}_{k|k-1}^T\} = \mathbf{P}_{k|k-1}$$

Augmented measurement vector

$$\begin{bmatrix} \hat{\boldsymbol{\theta}}_{k|k-1} \\ \bar{\mathbf{d}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{12} \\ \mathbf{H} \end{bmatrix} \boldsymbol{\theta}_k + \begin{bmatrix} \mathbf{e}_{k|k-1} \\ \bar{\mathbf{n}}_k \end{bmatrix}$$

Kalman filter: weighted LS formulation

(Unconstrained) KF is obtained by solving the WLS problem

$$\hat{\boldsymbol{\theta}}_k = \arg\min_{\boldsymbol{\theta}_k} \left\|\mathbf{P}_{k|k-1}^{-1/2}(\hat{\boldsymbol{\theta}}_{k|k-1} - \boldsymbol{\theta}_k)\right\|_2^2 + \|\bar{\mathbf{d}}_k - \mathbf{H}\boldsymbol{\theta}_k\|_2^2$$

Allows us to add constraints on the state variables.

$$\hat{\boldsymbol{\theta}}_k = \hat{\boldsymbol{\theta}}_{k|k-1} + \mathbf{K}_k(\bar{\mathbf{d}}_k - \mathbf{H}\hat{\boldsymbol{\theta}}_{k|k-1})^{-1} \quad \textit{(KF update equation)}$$

$$\mathbf{P}_k = (\mathbf{I}_{12} - \mathbf{K}_k\mathbf{H})\mathbf{P}_{k|k-1} \quad \textit{(Estimated covariance)}$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}^T(\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T + \mathbf{I})^{-1}$$

Recall $\hat{\boldsymbol{\Theta}}_k = \mathrm{vec}^{-1}(\hat{\boldsymbol{\theta}}_k) = \left[\begin{array}{c|c} \hat{\mathbf{Q}}_k & \hat{\mathbf{t}}_k \end{array}\right]$.

As earlier, project the solution onto the constraint set

$$\hat{\mathbf{Q}}_{\mathrm{UC},k} = (\hat{\mathbf{Q}}_k\hat{\mathbf{Q}}_k^T)^{-1/2}\hat{\mathbf{Q}}_k.$$

Remarks

Unitarily constrained least-squares

$$\arg\min_{\boldsymbol{\theta}} \quad \|\bar{\mathbf{d}} - \mathbf{H}\boldsymbol{\theta}\|_2^2$$
$$\text{s.to} \quad \mathbf{Q}_k \in \mathcal{V}_{3,3}$$

Unitarily constrained Kalman filter

$$\arg\min_{\boldsymbol{\theta}_k} \left\|\mathbf{P}_{k|k-1}^{-1/2}(\hat{\boldsymbol{\theta}}_{k|k-1} - \boldsymbol{\theta}_k)\right\|_2^2 + \|\bar{\mathbf{d}}_k - \mathbf{H}\boldsymbol{\theta}_k\|_2^2$$
$$\text{s.to} \quad \mathbf{Q}_k \in \mathcal{V}_{3,3}$$

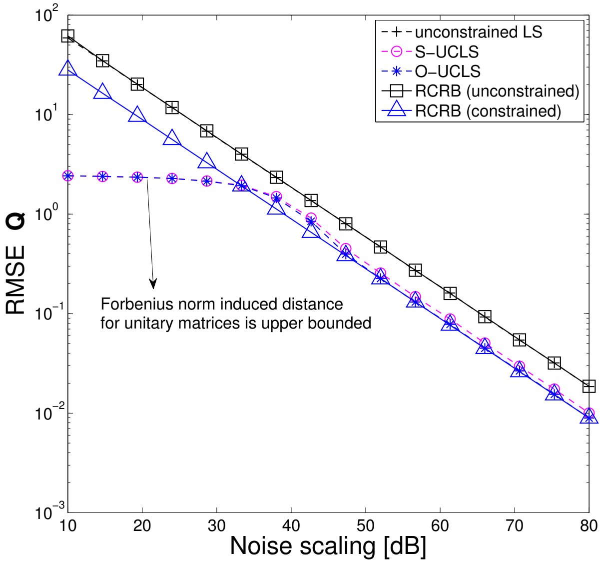
Transform the constrained problem into an unconstrained one using:

1. the matrix exponential $\exp(\mathbf{X})$
2. or the Cayley transform $(\mathbf{I} - \mathbf{X})(\mathbf{I} + \mathbf{X})^{-1}$

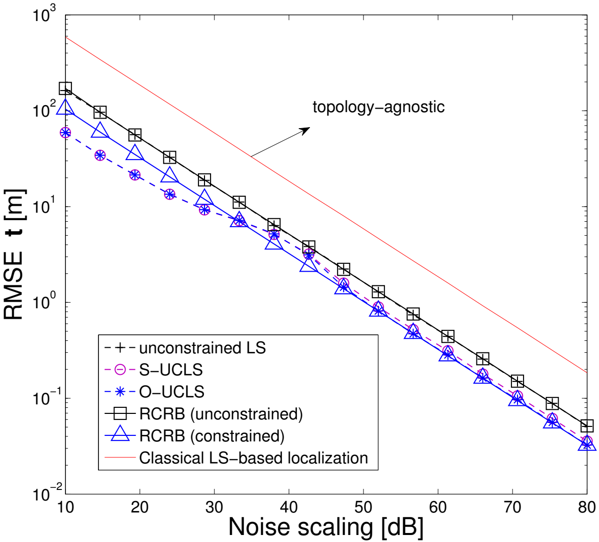
then use Newtons method.

Simulations: static

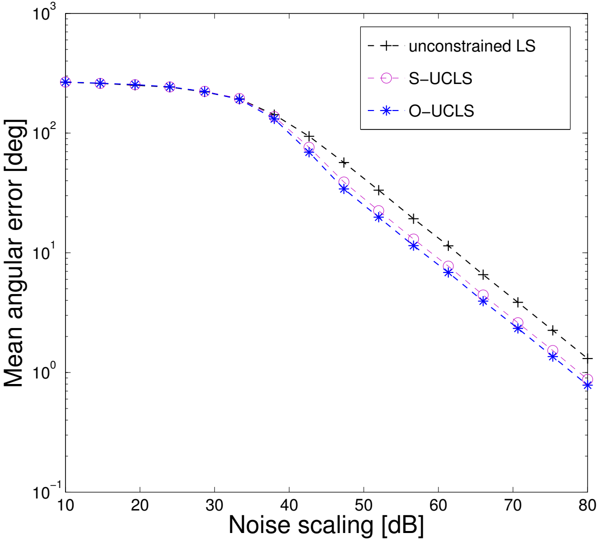
$N = 5$ sensors mounted on a rectangle based pyramid, $M = 4$ anchors



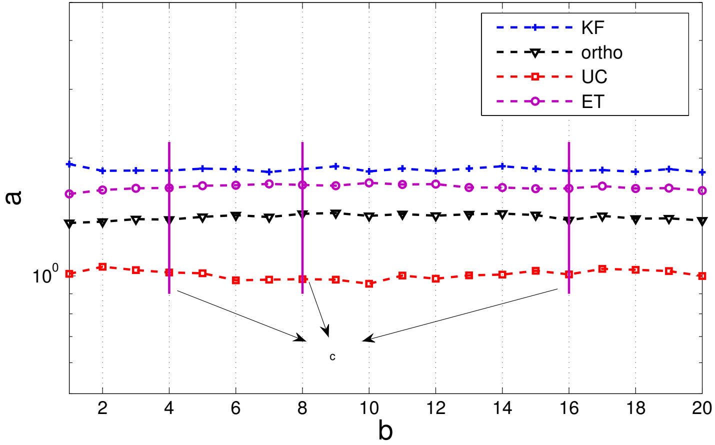
Simulations: static



Simulations: static



Simulations: dynamic



Thank you!

Appendix: Unitarily constrained LS

Recall $\bar{\mathbf{D}} = \bar{\mathbf{A}}\mathbf{S} + \bar{\mathbf{N}}$ and $\mathbf{S} = \mathbf{Q}\mathbf{C} + \mathbf{t}\mathbf{1}_N^T$

Decouple $\mathbf{Q}$ and $\mathbf{t}$

$$\mathbf{S}\mathbf{U}_N = \mathbf{Q}\mathbf{C}\mathbf{U}_N$$

$$\mathbf{P}_N = \mathbf{I}_N - \tfrac{1}{N}\mathbf{1}_N\mathbf{1}_N^T = \mathbf{U}_N\mathbf{U}_N^T$$

Use $\mathbf{S}\mathbf{U}_N$ to get

$$\tilde{\mathbf{D}} = \bar{\mathbf{A}}\mathbf{Q}\bar{\mathbf{C}} + \tilde{\mathbf{N}} \quad \Rightarrow \quad \tilde{\mathbf{d}} = (\bar{\mathbf{C}}^T \otimes \bar{\mathbf{A}})\mathbf{q} + \tilde{\mathbf{n}}$$

Unitarily Constrained Least-Squares

$$\begin{aligned}
\hat{\mathbf{Q}}_{\mathrm{UCLS}} =& \arg\min_{\mathbf{Q}} \|\tilde{\mathbf{d}} - (\bar{\mathbf{C}}^T \otimes \bar{\mathbf{A}})\mathbf{q}\|_2^2 \\
& \mathrm{s.to} \quad \mathbf{q} = \mathrm{vec}(\mathbf{Q}),\ \mathbf{Q}^T\mathbf{Q} = \mathbf{I}_3, \\
\hat{\mathbf{t}}_{\mathrm{UCLS}} =& \min_{\mathbf{t}} \quad \|\bar{\mathbf{D}} - \bar{\mathbf{A}}(\hat{\mathbf{Q}}_{\mathrm{S-UCLS}}\mathbf{C} + \mathbf{t}\mathbf{1}_N^T)\|_F^2 \\
=& \frac{1}{N}(\bar{\mathbf{A}}^\dagger\bar{\mathbf{D}} - \hat{\mathbf{Q}}_{\mathrm{UCLS}}\mathbf{C})\mathbf{1}_N.
\end{aligned}$$