Modelling Bandlimited Signals in Self-Dual Differential Equations

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***Abstract*—Bandlimited signals and differential equations play fundamental roles in signal processing and communications, yet their relationship remains largely unexplored. In this work, we investigate how bandlimitedness emerges in the solutions to a class of differential equations that exhibit time-domain and frequency-domain self-duality. We derive the necessary and sufficient conditions in time and frequency domains under which solutions of such equations are bandlimited. Through illustrative examples of well-known bandlimited signals and their associated differential equations, we elaborate on the realization of the derived conditions. Furthermore, we demonstrate how bandlimited behavior can be induced in otherwise non-bandlimited systems by introducing suitable differential operators. We briefly discuss a potential research direction on the applications of such differential equations in generative models as well. However, the main purpose of this work is to present the analytical results.**

***Index Terms*—Bandlimited signals, differential equations.**

I. Introduction

Differential equations are indispensable tools for describing signals [1], [2], analyzing communication channels [3], and describing system dynamics [4], [5], [6]. By capturing how signals evolve over time or space, these equations provide a mathematical framework for understanding system behavior and designing effective signal processing techniques. Differential equations also serve as the fundamental backbone of various generative models, which often aim to generate novel and meaningful signals, such as images and audio, from a random noise sample [7], [8].

In a recent study, Ghule *et al.* [9] characterized a rather interesting class of differential equations that demonstrate self-duality in time and frequency domains. In this context, "self-duality" refers to the property that certain differential equations preserve their structural form under transformation between these domains. An ordinary differential equation (ODE), $f\left(t, x(t), \frac{\mathrm{d}x(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}, ..., \frac{\mathrm{d}^nx(t)}{\mathrm{d}t^n}\right) = 0$, is said to be *strictly self-dual* in time and frequency domains if the Fourier transform ($\mathcal{F}$) satisfies:

$$\begin{aligned}&\mathcal{F}\left\{f\left(t, x(t), \frac{\mathrm{d}x(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}, ..., \frac{\mathrm{d}^nx(t)}{\mathrm{d}t^n}\right)\right\}\\ &= f\left(\Omega, X(\Omega), \frac{\mathrm{d}X(\Omega)}{\mathrm{d}\Omega}, \frac{\mathrm{d}^2X(\Omega)}{\mathrm{d}\Omega^2}, ..., \frac{\mathrm{d}^nX(\Omega)}{\mathrm{d}\Omega^n}\right).\end{aligned} \tag{1}$$

TABLE I
Bandlimited signals and associated weakly self-dual ODEs.

Common Name	Signal (in time)	Bandwidth (in Ω)	Differential Equation (in time)
Sinc	$(\sin t)/t$	$(-1, 1)$	$tx''(t) + 2x'(t) + tx(t) = 0$
Bessel Functions [10] (First Kind)	$J_\alpha(t)$ $\alpha \in \mathbb{Z}$	$(-1, 1)$	$t^2x''(t)+tx'(t)+(t^2-\alpha^2)x(t) = 0$
Prolate Spheroidal Wave Functions [11]	$\psi_n(t; c)$	$(-c, c)$	$(1-t^2)x''(t) - 2tx'(t) + (\lambda_n - c^2t^2)x(t) = 0$

A relaxed notion of *weakly self-dual* ODEs exists in the following form:

$$\sum_{i=0}^{n} p_i(t)x^{(i)}(t) = 0, \tag{2}$$

where $x^{(i)}(t)$ is the i$^{\text{th}}$ derivative of $x(t)$, $x^{(0)}(t) = x(t)$, and $p_i(t) \forall\, 0 \leq i \leq n$ is a polynomial in t of degree at most m with $p_n(t) \neq 0$. In the frequency domain, the differential equation shows a similar form, which can be expressed as,

$$\sum_{i=0}^{m} Q_i(\Omega)X^{(i)}(\Omega) = 0, \tag{3}$$

where $Q_i(\Omega) \forall\, 0 \leq i \leq m$ is a polynomial of degree at most n with $Q_m(t) \neq 0$.

It is interesting to note that many widely studied signals associated with bandlimitedness are solutions to weakly self-dual ODEs. Selected examples are shown in Table I. Motivated by this, we aim to explore the manifestation of bandlimitedness in the solutions of ODEs, especially of the weakly self-dual form. The contributions of this work are summarized below.

1) We present an exhaustive study of weakly self-dual differential equations and their bandlimited solutions.
2) We devise necessary and sufficient conditions for ODEs to generate bandlimited solutions.
3) We propose a novel approach to model bandlimited solutions in arbitrary ODEs.
4) We study and demonstrate, via illustrative examples, the application of the devised theory.

To the best of our knowledge, no prior work has addressed the analysis of bandlimited solutions of differential equations.

In this study, we limit our analysis to only real functions of time. Our work primarily focuses on the analytical results. The potential applications to generative models are speculative and may serve as future research directions.

II. Preliminaries

Bandlimited Signals. Bandlimitedness, as a property, is fundamental to the fields of signal processing and communications engineering. A signal $x(t), t \in \mathbb{R}$ is said to be bandlimited if $\exists$ a constant $B \geq 0$ such that the Fourier transform $X(\Omega) = \mathcal{F}\{x(t)\}, \Omega \in \mathbb{R}$ satisfies $X(\Omega) = 0, \forall |\Omega| \geq B$ [1].

Real Analytic Functions. A function f, with domain $\mathcal{D} \subseteq \mathbb{R}$ and range either $\mathbb{R}$ or $\mathbb{C}$, is real analytic at x_0 if f has a convergent power series representation on some interval of positive radius centred at x_0: $f(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j$. The function is said to be real analytic on $\mathcal{U} \subseteq \mathcal{D}$ if it is real analytic at each $x_0 \in \mathcal{U}$ [12].

Identity Theorem. Given f and g be real analytic functions on an open interval $\mathcal{D}$ and $\exists$ an open set $\mathcal{A} \subseteq \mathcal{D}$ such that $f(x) = g(x)\ \forall x \in \mathcal{A}$, then $f(x) = g(x)\ \forall x \in \mathcal{D}$ [12].

III. Main Results

In this section, we will study the various properties linked with bandlimited solutions of weakly self-dual ODEs.

A. Necessity of Non-Analyticity

Our primary result can be framed in the theorem below.

Theorem 1. *Consider the linear differential equation $\sum_{i=0}^{n} p_i(t) u^{(i)}(t) = 0$, characterized by the weakly self-dual property, where each $p_i(t)$ is a polynomial. A non-trivial solution $u(t) = x(t)$ is bandlimited with bandwidth Ω_0 if and only if its Fourier transform $X(\Omega) = \mathcal{F}\{x(t)\}$ is non-analytic on the boundary of its support, specifically at $\Omega = \pm\Omega_0$.*

Proof. Consider the general weakly self-dual linear ODE in time:

$$\sum_{i=0}^{n} p_i(t) u^{(i)}(t) = 0, \tag{4}$$

where $p_i(t)$ is a polynomial of degree at-most m and $u^{(i)}$ is the i$^{\text{th}}$ derivative of $u(t)$. Let $x(t)$ be a bandlimited function such that $u(t) = x(t)$ is a solution to (4). Looking at the Fourier domain, due to weak self-duality, (4) translates to the following:

$$\sum_{i=0}^{m} Q_i(\Omega) U^{(i)}(\Omega) = 0, \tag{5}$$

where $Q_i(\Omega)$ is a polynomial of degree at-most n and $U^{(i)}$ is the i$^{\text{th}}$ derivative of $U(\Omega)$. Clearly, $U(\Omega) = X(\Omega)$ is a solution to (5), where $X(\Omega)$ is the Fourier transform of $x(t)$. For $X(\Omega)$ to be bandlimited, $\exists\, \Omega_0$ such that $\Omega_0 = \sup\{\, |\Omega| : X(\Omega) \neq 0 \,\}$ and $X(\Omega) = 0, \forall\, |\Omega| > \Omega_0$.

Case 1. Let us begin by assuming that the bandlimited solution $X(\Omega)$ is well-defined and real analytic $\forall \Omega \in \mathbb{R}$. Now, consider a second function $G(\Omega) = 0\ \forall \Omega$. It can be clearly seen that $G(\Omega)$ is real analytic and $X(\Omega) = G(\Omega)\ \forall |\Omega| > \Omega_0$. However, from the Identity Theorem, we have the following:

$$X(\Omega) = 0 \quad \forall \Omega \in \mathbb{R}. \tag{6}$$

Thus, non-trivial functions real analytic on $\mathbb{R}$ can *not* be bandlimited. Now it remains to demonstrate that for a non-trivial solution, the band transition points must be non-analytic. For the scope of this work, we denote all points of non-analyticity as singular points or singularities.

Case 2. Consider the case of a non-trivial bandlimited solution $x(t)$ and $X(\Omega) = \mathcal{F}\{x(t)\}$. Let $\mathcal{S}_X$ be the set of all singular points for $X(\Omega)$ in $\mathbb{R}$. Note that $X(\Omega)$ remains analytic between any two consecutive singular points. Following the arguments in Case 1, the bandwidth Ω_0 satisfies the following:

$$\Omega_0 \leq \min\{|\Omega_s| : \Omega_s \in \mathcal{S}_X \text{ and } X(\Omega) = 0\, \forall |\Omega| > \Omega_s\}. \tag{7}$$

Moreover, since $X(\Omega)$ is piecewise analytic, between any two consecutive singularities, if $X(\Omega)$ assumes a non-zero value at any point, it cannot be identically zero on any open interval between the singularities (consequence of the Identity Theorem). Thus, Ω_0 satisfies the following:

$$\Omega_0 \geq \min\{|\Omega_s| : \Omega_s \in \mathcal{S}_X \text{ and } X(\Omega) = 0\, \forall |\Omega| > \Omega_s\}. \tag{8}$$

From (7) and (8), we conclude that $\Omega = \pm\Omega_0$ are necessarily singular points. □

From the above proof, the following remarks can be made.

Remark. *The singularities of the solution $X(\Omega)$ can only occur at the zeroes of the polynomial coefficients $Q_i(\Omega)$ in (5), and more strictly, at the zeroes of the coefficient of the highest order derivative $Q_m(\Omega)$.*

Remark. *For a weakly self-dual linear ODE of form (4), let $x(t)$ be a solution with bandwidth Ω_0. If $\mathcal{S}_X$ be the set of all singularities of $X(\Omega)$, the bandwidth satisfies the following:*

$$\Omega_0 \leq \max_{\Omega_s \in \mathcal{S}_x} |\Omega_s|. \tag{9}$$

Theorem 2. *For any $X(\Omega)$ satisfying a linear differential equation of the form (5), where $Q_i(\Omega)$ is real analytic on $\mathbb{R}$ and not necessarily a polynomial $\forall i$, is bandlimited by Ω_0 if and only if $\pm\Omega_0$ are singularities of $X(\Omega)$.*

The proof of Theorem 1 trivially extends to Theorem 2.

B. Analysis of Special Cases

In this section, we analyze two types of time-domain differential equations that yield bandlimited solutions due to singularities in the frequency domain. These two types exhaustively cover all the weakly self-dual ODEs of form (4) with the potential of bandlimited solutions.

Type I. Consider a differential equation in the Fourier domain with singularities at Ω_1 and Ω_2 of the following form:

$$(\Omega - \Omega_1)(\Omega - \Omega_2)\, \hat{\mathcal{L}}(X(\Omega)) = 0, \tag{10}$$

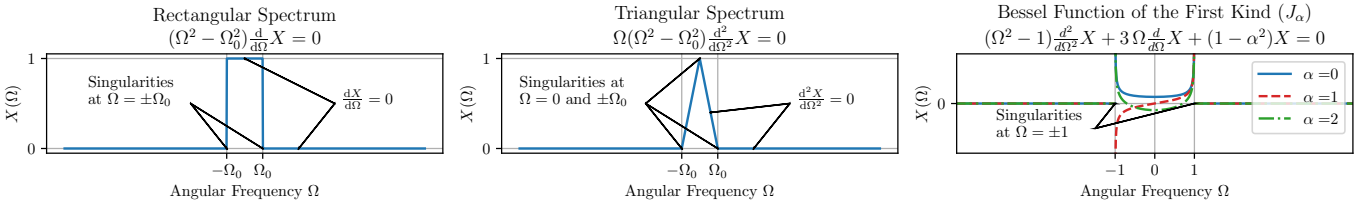

Fig. 1. Frequency-domain representations of rectangular spectrum $\mathrm{rect}(\Omega/2\Omega_0)$, triangular spectrum $\mathrm{tri}(\Omega/\Omega_0)$, and Bessel function of the first kind J_α (of order α) respectively (from left to right) and their associated differential equations show that singularities support the bandlimitedness.

where $\hat{\mathcal{L}}$ is a differential operator such that $\hat{\mathcal{L}}(X(\Omega))=0$ is well-behaved at $\Omega=\Omega_1,\Omega_2$. Hence, we have,

$$\left(\Omega^2-(\Omega_1+\Omega_2)\Omega+\Omega_1\Omega_2\right)\,\hat{\mathcal{L}}(X(\Omega))=0.$$

Taking the inverse Fourier transform gives us,

$$\left(-\partial_t^2+j(\Omega_1+\Omega_2)\partial_t+\Omega_1\Omega_2\right)\,\mathcal{L}(x(t))=0,$$

where operator $\partial_t=\frac{\mathrm{d}}{\mathrm{d}t}$ and $\mathcal{L}(x(t))=\mathcal{F}^{-1}\{\hat{\mathcal{L}}(X(\Omega))\}$ which may or may not be a differential expression. Substituting $\Omega_1=-\Omega_2=\Omega_0$, $(\Omega^2-\Omega_0^2)\,\hat{\mathcal{L}}(X(\Omega))=0$, we have,

$$(\partial_t^2+\Omega_0^2)\,\mathcal{L}(x(t))=0. \tag{11}$$

Note that, if $\hat{\mathcal{L}}(X(\Omega))$ is of the weakly self-dual form with polynomial coefficients of degree at most $n-2$, (10) will be a weakly self-dual ODE with polynomial coefficients of degree at most n. The result in (11) shows that the presence of the operator $(\partial_t^2+\Omega_0^2)$ in the differential equation in the time-domain is a *sufficient* condition to attain singularities at $\pm\Omega_0$, allowing for a solution bandlimited by $|\Omega_0|$. For some constants C_1 and C_2, the general solution of (11) is given by,

$$\mathcal{L}(x(t))=C_1\cos(\Omega_0 t)+C_2\sin(\Omega_0 t). \tag{12}$$

Type II. Consider a general weakly self-dual ODE in the Fourier domain with singularities at $\pm\Omega_0$ as follows:

$$\sum_{i=0}^{r}A_i(\Omega)X^{(i)}(\Omega)+(\Omega^2-\Omega_0^2)\sum_{i=r+1}^{m}B_i(\Omega)X^{(i)}(\Omega)=0, \tag{13}$$

$\forall i$, $A_i(\Omega)$ is a polynomial of degree at most n and $B_i(\Omega)$ is a polynomial of degree at most $n-2$. Taking the inverse Fourier transform, due to the weak self-duality, (13) becomes,

$$\sum_{i=0}^{n}p_i(t)x^{(i)}(t)+(\partial_t^2+\Omega_0^2)\sum_{i=0}^{n-2}q_i(t)x^{(i)}(t)=0, \tag{14}$$

where $\forall i$, $p_i(t)$ is a polynomial of degree at most r and $q_i(t)$ is a polynomial of degree at most m. Note that,

$$\begin{aligned}
&\partial_t^2\left(\sum_{i=0}^{n-2}q_i(t)x^{(i)}(t)\right)\\
&=\sum_{i=0}^{n-2}\left(q_i(t)x^{(i+2)}(t)+2q_i'(t)x^{(i+1)}+q_i''(t)x^{(i)}(t)\right)\\
&=\sum_{i=0}^{n-2}q_i''(t)x^{(i)}(t)+\sum_{i=1}^{n-1}2q_{i-1}'(t)x^{(i)}+\sum_{i=2}^{n}q_{i-2}(t)x^{(i)}.
\end{aligned}$$

Since q_i is only defined for $0\le i\le n-2$, we extend the definition by defining $q_i(t)=0\ \forall i\in\{-2,-1,n-1,n\}$. Thus, substituting these values in (14), we get the following:

$$\sum_{i=0}^{n}\Big\{\big(p_i(t)+q_{i-2}(t)+2q_{i-1}'(t)+(\partial_t^2+\Omega_0^2)\,q_i(t)\big)x^{(i)}(t)\Big\}=0, \tag{15}$$

where $p_i(t)$ is a polynomial of degree at most r and $q_i(t)$ is a polynomial of degree at most m.

Remark. *Note that the weakly self-dual ODEs of Type I are special cases of Type II ODEs, with $r+1=0$.*

IV. Some Illustrative Examples

To better understand the implications of the results obtained in Section III, we consider a few illustrative examples of well-known bandlimited functions that highlight how the proposed framework operates in specific cases. Figure 1 also shows the frequency-domain representations of some of these examples for visualization.

Example 1 (Rectangular Spectrum or Sinc). Consider the sinc function which is bandlimited between $\pm\Omega_0$ as,

$$x(t)=\frac{\sin(\Omega_0 t)}{t}. \tag{16}$$

It satisfies the following differential equation:

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\{t\,x(t)\}=-\Omega_0^2\,t\,x(t)\quad\Big\{=-\Omega_0^2\sin(\Omega_0 t)\Big\}\\
&\implies 2\frac{\mathrm{d}x(t)}{\mathrm{d}t}+t\,\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}=-\Omega_0^2\,t\,x(t),
\end{aligned} \tag{17}$$

which is a weakly self-dual linear differential equation. Note that (17) satisfies the form (11) discussed in Type I. Transforming (17) to the frequency-domain gives the following:

$$(\Omega^2-\Omega_0^2)\frac{\mathrm{d}X(\Omega)}{\mathrm{d}\Omega}=0. \tag{18}$$

A valid solution to (18) must satisfy the following:

$$\text{either, }\frac{\mathrm{d}X(\Omega)}{\mathrm{d}\Omega}=0\text{ or, }\Omega^2-\Omega_0^2=0. \tag{19}$$

The above conditions imply that the rectangular spectrum remains constant over its bandwidth. The derivative, $\frac{\mathrm{d}X(\Omega)}{\mathrm{d}\Omega}$, produces impulses at $\Omega=\pm\Omega_0$, which are accounted for by the singularities caused by the factor $\Omega^2-\Omega_0^2=0$.

Example 2 (Triangular Spectrum or Sinc^2). Consider the squared sinc function with the triangular spectrum as,

$$X(\Omega) = \operatorname{tri}\left(\frac{\Omega}{\Omega_0}\right) = \begin{cases} 1 - \dfrac{|\Omega|}{\Omega_0}, & |\Omega| < \Omega_0 \\ 0, & |\Omega| > \Omega_0 \end{cases}. \tag{20}$$

In the frequency domain, it satisfies the following ODE:

$$\Omega(\Omega^2 - \Omega_0^2)\frac{\mathrm{d}^2 X(\Omega)}{\mathrm{d}\Omega^2} = 0. \tag{21}$$

A solution to this ODE must yield the following:

$$\text{either } \Omega = 0, \Omega = \pm\Omega_0, \text{ or } \frac{\mathrm{d}^2 X(\Omega)}{\mathrm{d}\Omega^2} = 0, \tag{22}$$

which corresponds to the singular points ($\Omega = 0, \pm\Omega_0$) and the linear spectrum within the band $|\Omega| < \Omega_0$, respectively. Note that (21) satisfies the form (10) discussed in Type I.

Example 3 (Bessel ODE). Consider the Bessel ODE [10]:

$$t^2\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + t\frac{\mathrm{d}x(t)}{\mathrm{d}t} + (t^2 - \alpha^2)x(t) = 0. \tag{23}$$

Transforming it to the frequency domain, we have,

$$\frac{\mathrm{d}^2}{\mathrm{d}\Omega^2}\left[\Omega^2 X(\Omega)\right] - \frac{\mathrm{d}}{\mathrm{d}\Omega}\left[\Omega X(\Omega)\right] - \frac{\mathrm{d}^2 X(\Omega)}{\mathrm{d}\Omega^2} - \alpha^2 X(\Omega) = 0$$

$$\implies (\Omega^2 - 1)\frac{d^2 X}{d\Omega^2} + 3\,\Omega\frac{dX}{d\Omega} + (1 - \alpha^2)X = 0. \tag{24}$$

In the above equation, a singularity occurs at $\Omega = \pm 1$ as the coefficient of the double derivative term vanishes. This allows the standard Bessel functions to be bandlimited between $\Omega \in (-1, 1)$. It can be shown that (23) follows the expression (15) by a simple rearrangement of coefficients.

Example 4 (Angular Prolate Spheroidal Functions). For real values of c, consider the differential equation [11]:

$$(1 - t^2)\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \left(\chi_0 - c^2 t^2\right) y(t) = 0 \tag{25}$$

This equation has solutions that remain continuous on the closed interval $t \in [-1, 1]$ only for a discrete sequence of positive real values of the parameter χ_0. For each eigenvalue $\chi_n(c)$, where $n = 0, 1, 2, \ldots$, there exists a unique solution $S_{0n}(c, t)$, which are called *angular prolate spheroidal functions*. Taking the Fourier transform of (25) gives the following:

$$\left(c^2 - \Omega^2\right)\frac{\mathrm{d}^2 Y(\Omega)}{\mathrm{d}\Omega^2} - 2\Omega\frac{\mathrm{d}Y(\Omega)}{\mathrm{d}\Omega} + \left(\chi_0 - \Omega^2\right) Y(\Omega) = 0. \tag{26}$$

The above ODE has singularities at $\Omega = \pm c$ (the frequency parameter), allowing bandlimited solutions. Moreover, it may be noted that the equations in time (25) and frequency (26) domains show a striking resemblance and show a stronger self-duality (although not strict) than the rest of the considered examples.

V. Applications and Future Directions

Bandlimited solutions of differential equations find many applications, from exploring a system to designing one. In this section, we will discuss two such applications — designing bandlimited systems from differential equations and exploring the inherent bandlimitedness in generative models.

A. Controlled Induction of Bandlimitedness

The operator $\left(\partial_t^2 + \Omega_0^2\right)$ derived in (11) provides us with the ability to describe a system with bandlimited solutions using differential equations which would not have produced bandlimited solutions otherwise. Examples in Section IV demonstrate how the presence of this operator leads to bandlimited solutions. In the discussion that follows, we will start with an ODE describing a non-bandlimited function and demonstrate how this operator can induce bandlimitedness.

Example. Consider the differential equation below describing a Gaussian function $x(t) = Ke^{-t^2/2}$ for an arbitrary constant K as,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + tx(t) = 0. \tag{27}$$

It can be shown that any non-trivial solution of (27) cannot be bandlimited. Note that the Fourier transform of a Gaussian is a Gaussian. We aim to bandlimit the function between $\pm\Omega_0$. Thus, in (27), operating with $\left(\partial_t^2 + \Omega_0^2\right)$ on both the sides, we have,

$$x'''(t) + tx''(t) + (2 + \Omega_0^2)x'(t) + \Omega_0^2 tx(t) = 0. \tag{28}$$

From (12), the general solution of (28) is given by,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + tx(t) = C_1\cos(\Omega_0 t) + C_2\sin(\Omega_0 t), \quad C_1, C_2 \in \mathbb{R}. \tag{29}$$

Note that, at $C_1 = 0$ and $C_2 = K\sqrt{2\pi}e^{-\Omega_0^2/2}$, the solution $x(t) = Ke^{-t^2/2} * \operatorname{sinc}(\Omega_0 t)$ satisfies (29), which is the desired bandlimited Gaussian function, where $*$ denotes convolution.

The operator $\left(\partial_t^2 + \Omega_0^2\right)$ inherently models bandlimitedness through a differential equation by adding two degrees of freedom in the system (C_1 and C_2). A similar analysis can be carried out for various other systems whenever bandlimited solutions are required from differential equations.

B. A Potential Future Direction in Sampling Theory and Generative Models

A non-trivial manifestation of bandlimited solutions of differential equations can be seen in various modern generative models. Generative models such as diffusions [7] and flow-matching [13] models rely on neural networks to learn and solve ODEs [7]. The probability-flow ODE for score-based diffusion, derived by Song *et al.* [7] is given by,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x, t) - \frac{1}{2}g^2(t)\nabla_x \log p_t(x). \tag{30}$$

where $x(t)$ is the function modelling the data state in the forward process (noise addition) and $\nabla_x \log p_t(x)$ is the marginal probability predicted by the underlying neural network. In deterministic models, such as denoising diffusion implicit models (DDIMs) [14], the underlying ODEs take a deterministic form.

The generation trajectory $x(t)$ is solved for via discretizations, and the intermediate generations can be considered the samples of $x(t)$. Recall that sampling can only be performed faithfully, without the risk of aliasing, for a bandlimited function [1]. Although the ODEs in generative models are not designed to explicitly enforce bandlimitedness, we empirically

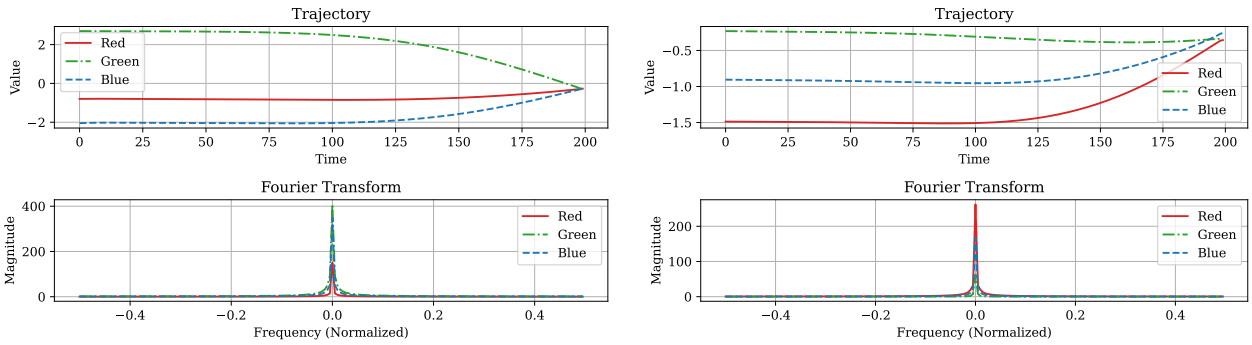

Fig. 2. The time series capturing the development in the values (trajectory) of a pixel in a DDIM generation shows bandlimited behavior. The plots show the trajectories corresponding to the red, green, and blue channels of two independent pixels (left and right) and their corresponding Fourier transforms.

observed that the generation trajectories of many modern models [13], [14] are extremely low-pass and nearly bandlimited. Figure 2 shows the (1) time evolution and (2) the Fourier transform of the generation trajectories of the two pixels of images generated using a DDIM model trained on 32×32 CIFAR-10 images [15].

The above observation implies that the (deterministic) generative models learn to incorporate bandlimitedness on their own. Learning to model such properties also costs additional resources during training. We conjecture that modelling bandlimitedness in their guiding ODEs by appropriately designing the neural network and constraining the noise schedule can lead to more efficient models. Future work may include a detailed development of the above.

VI. Conclusion

In this study, we investigated the relationship between bandlimited signals and differential equations in both the time and frequency domains. Our results show that singularities in a frequency-domain linear differential equation are necessary for the existence of bandlimited solutions. We further demonstrated that non-trivial real-analytic solutions of such differential equations cannot be bandlimited. Extending these findings, we analyzed the weakly self-dual class of differential equations and derived a sufficient condition under which time-domain differential equations admit bandlimited solutions. We also showed how this condition can be leveraged to induce bandlimitedness in differential equations whose solutions are otherwise not bandlimited. Finally, we discussed the relevance of differential equations with bandlimited solutions in generative modeling and outlined potential directions for future research.

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