Statistical Fidelity, Computational Cost, and Moment-Matched Surrogates for FSO Turbulence Channels

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***Abstract*—Atmospheric turbulence induces random irradiance fluctuations that degrade the reliability of free-space optical (FSO) communication links. Accurate statistical modeling of these fluctuations is essential for predicting bit error rate, outage probability, and ergodic capacity. This paper presents a systematic comparison of the lognormal, Gamma-Gamma, and Málaga irradiance models with respect to statistical fidelity and computational cost. A method-of-moments (MM) construction derives a lognormal surrogate matched to the first two moments of each reference distribution. The models are evaluated using PDF mean-squared error, the Kolmogorov distance between CDFs, and complementary CDF tail accuracy, alongside empirical runtime benchmarking. The Málaga model achieves the highest accuracy across turbulence regimes; the Gamma-Gamma model provides a practical balance between fidelity and computational efficiency; and the lognormal surrogate offers a rapid, low-cost alternative well suited to weak turbulence and large-scale parameter screening.**

***Index Terms*—Optical communication, atmospheric turbulence, irradiance, scintillation, Málaga distribution, Gamma-Gamma distribution, lognormal fading, method of moments.**

I. Introduction, Related work and Preliminaries

Free-space optical (FSO) communication has emerged as a promising technology for high-capacity wireless links owing to its large bandwidth, license-free spectrum, low electromagnetic interference, and rapid deployment [1]–[4]. These attributes make FSO systems attractive for last-mile connectivity, inter-building links, airborne platforms, and satellite-to-ground communication.

A major impairment in FSO links is atmospheric turbulence. Random refractive-index variations distort the optical wavefront, producing irradiance fluctuations (scintillation) at the receiver [2], [5]. Since performance metrics such as bit error rate (BER), outage probability, and ergodic capacity depend directly on the irradiance distribution, turbulence modeling plays a central role in FSO performance evaluation. Additional effects such as pointing errors and aperture averaging can further influence link reliability.

The lognormal distribution provides a simple approximation under weak turbulence but becomes less accurate as turbulence strength increases [5]. The Gamma-Gamma model, which represents irradiance as the product of large-scale and small-scale fluctuations, has become a standard choice for FSO performance analysis [6]. The Málaga distribution offers a more general framework encompassing several classical turbulence models, albeit at increased computational complexity due to finite summations and modified Bessel functions [7].

A. Related Work

Statistical modeling of atmospheric turbulence originates from Kolmogorov's theory of the inertial cascade and its extension to optical wave propagation by Tatarskii [5], [8]. The lognormal model was an early choice for weak turbulence, capturing multiplicative irradiance fluctuations while yielding simple analytical expressions [2], [9]. However, its accuracy deteriorates as turbulence strength increases. To address this limitation, composite models were introduced in which irradiance is represented through independent large-scale and small-scale fluctuation processes. The Gamma-Gamma model has demonstrated strong agreement with optical propagation measurements across moderate turbulence regimes and has been widely applied to BER, outage, and capacity analysis [6], [10]–[12]. Related effects such as pointing errors and aperture averaging have also been extensively studied [13].

The Málaga distribution provides a more general irradiance model with applicability across a wider range of turbulence conditions [7]. Despite its improved fidelity, several works continue to employ simpler models due to the computational burden of repeated evaluation of generalized distributions. While the Double Generalized Gamma distribution also spans a wide turbulence range [2], the present work focuses specifically on quantifying the fidelity-versus-cost trade-off and constructing a tractable moment-matched surrogate, a perspective not addressed in prior comparative studies.

While substantial literature exists on turbulence-induced fading models and their application to FSO performance analysis, a systematic assessment of statistical fidelity versus computational cost trade-off remains limited. This paper addresses that gap. The principal contributions are as follows:

- A unified comparison of the Lognormal (LN), Gamma-Gamma (GG), and Malaga irradiance models under a common simulation framework spanning representative turbulence conditions.
- A method-of-moments (MM) lognormal surrogate matched to the first two moments of the Gamma-Gamma and Malaga distributions. Since the first two

moments govern the average received optical power and scintillation strength, they provide a natural low-complexity basis for statistical approximation while avoiding the computational burden associated with higher-order matching.

Table I lists the principal notation used throughout this paper and positions the present work relative to prior studies.

TABLE I
KEY NOTATION AND COMPARISON WITH RELATED WORK (BELOW)

Symbol	Meaning
I	Received optical irradiance
I_L, I_S	Large- and small-scale turbulence components
α, β	Shape parameters (GG and Málaga)
ρ, Ω	Málaga coupling and coherence parameters
k	Summation index in the Málaga PDF
$K_\nu(\cdot)$	Modified Bessel function of the second kind, order ν
μ, σ^2	Lognormal log-mean and log-variance
C_n^2	Refractive-index structure parameter
σ_R^2	Rytov variance
ξ, γ', ω'	Málaga auxiliary quantities

Feature	Al-Habash [6]	Jurado-Navas [7]	Khalighi [10]	This work
Models compared	GG	Málaga	LN, GG	LN, GG, Málaga
MM surrogate	No	No	No	**Yes**
Cost analysis	No	No	No	**Yes**
Tail/CCDF metrics	No	No	Partial	**Yes**

B. Preliminaries: Atmospheric Turbulence

Atmospheric turbulence arises from random fluctuations in temperature and pressure, inducing spatially varying perturbations in the refractive index $n(\mathbf{r}, t)$ of air. Diffraction and interference along the propagation path convert the resulting phase distortions into random irradiance fluctuations at the receiver, quantified by the Rytov variance

$$\sigma_R^2 = 1.23\, C_n^2\, k^{7/6}\, L^{11/6}, \tag{1}$$

where $k = 2\pi/\lambda$ is the optical wavenumber, L is the propagation distance, and C_n^2 is the refractive-index structure parameter [2]. Weak turbulence ($\sigma_R^2 < 0.3$) produces mild scintillation for which lognormal approximations are generally adequate. Moderate turbulence ($0.3 \leq \sigma_R^2 \leq 1$) makes composite fading significant, and the Gamma-Gamma model is often effective. Strong turbulence ($\sigma_R^2 > 1$) produces heavy-tailed irradiance statistics for which generalized models such as Málaga are preferred.

II. STATISTICAL MODELS OF CHANNELS

This section presents the three irradiance distributions used in this work. The Gamma-Gamma model is introduced first as the principal composite-fading reference. The Málaga distribution is then presented as the generalized model it encompasses. Finally, the lognormal distribution is derived as a MM surrogate matched to the first two moments of a target distribution.

A. Gamma-Gamma Distribution

The Gamma-Gamma model represents irradiance I as the product of two statistically independent Gamma-distributed components, that is, $I = I_L\, I_S$, where I_L and I_S account for large-scale and small-scale irradiance fluctuations, respectively, with shape parameters $\alpha > 0$ and $\beta > 0$. Their individual PDFs are

$$f_{I_L}(x) = \frac{\alpha^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\alpha x}, \qquad x > 0, \tag{2}$$

$$f_{I_S}(x) = \frac{\beta^\beta}{\Gamma(\beta)}\, x^{\beta-1} e^{-\beta x}, \qquad x > 0. \tag{3}$$

The marginal irradiance PDF obtained from $I = I_L\, I_S$ is

$$f_I(I) = \frac{2\,(\alpha\beta)^{(\alpha+\beta)/2}}{\Gamma(\alpha)\,\Gamma(\beta)}\, I^{(\alpha+\beta)/2-1}\, K_{\alpha-\beta}\Big(2\sqrt{\alpha\beta I}\Big), \tag{4}$$

where $K_\nu(\cdot)$ is the modified Bessel function of the second kind of order ν.

Remark: Irradiance is normalized so that $\mathbb{E}[I] = 1$, a standard convention in FSO turbulence analysis that ensures a consistent comparison across statistical models.

B. Málaga Distribution

The Málaga model generalizes the Gamma-Gamma distribution by partitioning the scattered optical field into a coherent (line-of-sight) component and a diffuse component, thereby accommodating a wider range of propagation conditions [7]. When β is a positive integer, the irradiance PDF admits the finite-sum representation

$$f_I(I) = A \sum_{k=1}^{\beta} a_k\, I^{(\alpha+k)/2-1}\, K_{\alpha-k}\left(2\sqrt{\frac{\alpha\beta\, I}{\xi}}\right), \tag{5}$$

where the normalization constant A and coefficients a_k are

$$A = \frac{2\,\alpha^{\alpha/2}}{\Gamma(\alpha)\,(\gamma')^{1+\alpha/2}} \left(\frac{\gamma'\beta}{\xi}\right)^{\beta+\alpha/2}, \tag{6}$$

$$a_k = \frac{\binom{\beta-1}{k-1}\rho^{k-1}(1-\rho)^{\beta-k}}{(k-1)!}. \tag{7}$$

The auxiliary quantities γ', ω', and ξ are defined as

$$\gamma' = (1-\Omega)(1-\rho), \qquad \omega' = \Omega + (1-\Omega)\rho, \qquad \xi = \gamma'\beta + \omega', \tag{8}$$

where $\Omega \in [0, 1]$ is the fraction of power carried by the coherent component and $\rho \in [0, 1]$ governs the coupling between the coherent and diffuse scattered fields.

C. Lognormal Approximation via Method of Moments

Under weak turbulence, the log-irradiance is modeled as Gaussian, that is, $X = \ln I \sim \mathcal{N}(\mu, \sigma^2)$, yielding the lognormal irradiance PDF

$$f_I(I) = \frac{1}{I\,\sigma\sqrt{2\pi}}\, \exp\left(-\frac{(\ln I - \mu)^2}{2\sigma^2}\right), \qquad I > 0. \tag{9}$$

The mean and variance of a lognormal random variable are

$$\mathbb{E}[I] = e^{\mu+\sigma^2/2}, \quad \mathrm{Var}(I) = \left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}. \tag{10}$$

Given a target distribution with mean $m_1 = \mathbb{E}[I]$ and variance $v = \mathrm{Var}(I)$, inverting the expressions in (10) via MM yields

$$\sigma^2 = \ln\left(1+\frac{v}{m_1^2}\right), \qquad \mu = \ln(m_1) - \frac{\sigma^2}{2}. \tag{11}$$

These expressions allow the lognormal parameters to be matched exactly to the first two moments of the Gamma-Gamma or Málaga distribution, constructing a tractable surrogate for subsequent performance comparison.

III. Approximation Error Metrics

Accurate turbulence modeling requires that an approximating distribution agree with the reference not only over the bulk of the irradiance range but also in the tails, which govern outage and reliability measures. Specifically, we consider three complementary error measures: a PDF mean-squared error, a CDF supremum deviation, and a CCDF-based tail comparison. Below, we present details on these measures.

PDF Mean-Squared Error: The pointwise discrepancy between sampled PDFs is quantified by

$$\mathrm{MSE}_{\mathrm{PDF}} = \frac{1}{N}\sum_{i=1}^{N}\left(f_{\mathrm{ref}}(I_i) - f_{\mathrm{approx}}(I_i)\right)^2, \tag{12}$$

where $\{I_i\}_{i=1}^{N}$ is a uniform evaluation grid over the irradiance support. This metric penalizes squared deviations and is sensitive to local shape mismatch in high-probability regions.

Kolmogorov Distance: The maximum absolute deviation between CDFs provides a distribution-level measure of approximation error. This is computed as

$$D_K = \max_i \left|F_{\mathrm{ref}}(I_i) - F_{\mathrm{approx}}(I_i)\right|, \tag{13}$$

which corresponds to the empirical Kolmogorov-Smirnov statistic and converges to the theoretical supremum $\sup_I |F_1(I) - F_2(I)|$ as the grid is refined. Unlike $\mathrm{MSE}_{\mathrm{PDF}}$, this metric is bounded in $[0, 1]$ and is invariant to monotone reparameterization of the irradiance axis.

CCDF Tail Comparison: Outage probability in an FSO link is defined as $P_{\mathrm{out}} = P(I < I_{\mathrm{th}})$ for a given irradiance threshold I_{th}, and is therefore governed entirely by the lower tail of the irradiance distribution. A surrogate model that matches the bulk PDF accurately may still incur significant error in P_{out} if its tail decays at a different rate. The complementary CDF is therefore evaluated alongside the PDF and CDF to expose tail mismatch that scalar error metrics may not resolve.

IV. Simulation Methodology and Results

This section presents the numerical evaluation of the three irradiance models across a representative range of turbulence parameters. The simulation setup and grid definitions are described first. Subsections IV-A and IV-B present the Gamma-Gamma and Málaga comparisons, respectively, followed by a benchmark summary and discussion of the accuracy-versus-efficiency trade-off.

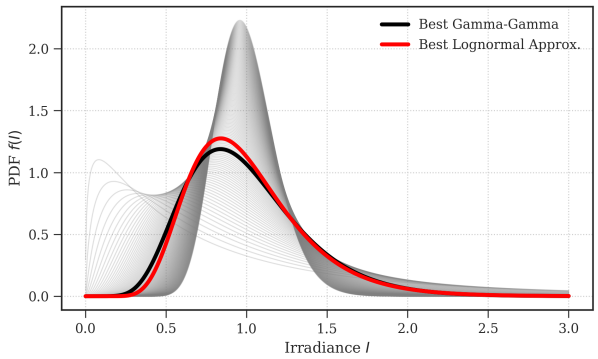

Fig. 1. PDF of the Gamma-Gamma model and its lognormal MM surrogate for representative (α, β) pairs. The surrogate matches the bulk shape accurately under moderate turbulence but diverges in the tails as α decreases.

Simulation Setup: All numerical evaluations were performed in Python 3.11 using NumPy and SciPy for special-function computation and probability density evaluation. The Málaga parameters were varied over the ranges $\alpha \in [2, 30]$, $\beta \in [4, 32]$, $\rho \in [0.1, 0.9]$, and $\Omega \in [0, 0.5]$. Irradiance was normalized to $\mathbb{E}[I] = 1$ throughout, consistent with the convention established in Section I-B. The Málaga distribution served as the reference in all comparisons, that is, $f_{\mathrm{ref}}(I) = f_{\mathcal{M}}(I)$, where $f_{\mathcal{M}}(I)$ denotes the Málaga irradiance PDF defined in (5). The evaluation grid comprised $N = 300$ uniformly spaced points over $I \in [0,\ 3.0]$ for PDF and CDF comparisons. Lognormal surrogate parameters were obtained via the MM expressions in (11).

Runtime Measurement: Each runtime measurement corresponds to evaluating a model PDF over 2000 irradiance grid points. Reported values are averages over multiple independent runs to reduce measurement variability and are obtained on an Apple M1 CPU with 16 GB RAM. These figures quantify comparative computational cost rather than hardware-independent absolute performance.

A. Gamma-Gamma Comparisons

Fig. 1 and 2 present the PDF, CDF, and CCDF of the Gamma-Gamma model against its lognormal MM surrogate across representative parameter settings. Figure 3 shows the approximation error over the full parameter grid.

The Gamma-Gamma model accurately captures the positive support and asymmetry of the irradiance fading distribution while remaining computationally tractable. As seen in Fig. 1, the lognormal surrogate matches the central body of the Gamma-Gamma PDF well under moderate turbulence, with the largest deviations appearing in the lower tail. Fig. 3 confirms that approximation error grows as α decreases, i.e., as large-scale fluctuations intensify, indicating that the lognormal surrogate becomes progressively less reliable in stronger turbulence regimes.

B. Málaga Comparisons

Fig. 4 and 5 present the Málaga PDF, CDF, and CCDF against the lognormal surrogate. The error heatmap in Fig. 6

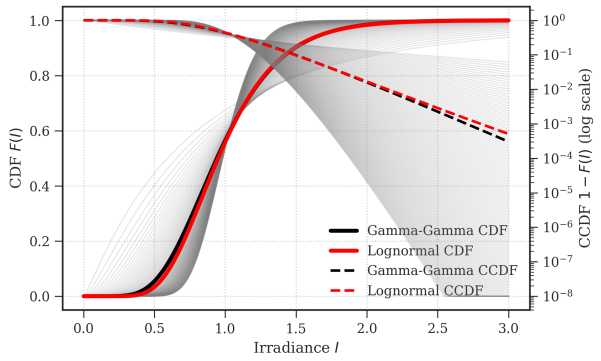

Fig. 2. CDF and CCDF of the Gamma-Gamma model and lognormal surrogate. Tail discrepancy in the CCDF is most pronounced at low irradiance values, directly affecting outage probability predictions.

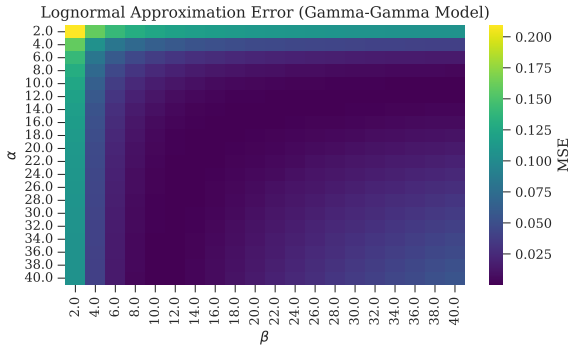

Fig. 3. Mean-squared error of the lognormal surrogate relative to the Gamma-Gamma model over the (α, β) parameter grid. Error increases systematically as α decreases, corresponding to stronger turbulence conditions.

spans the full $(\alpha, \beta, \rho, \Omega)$ parameter space.

The Málaga model provides the most flexible irradiance representation across the explored parameter ranges. Its advantage over simpler models is most pronounced when the distribution develops strong asymmetry or heavy tails, as occurs at low ρ and elevated Ω. In such regimes, as Fig. 5 shows, the lognormal surrogate correctly matches the first two moments but incurs significant error in the lower CCDF tail, which translates directly to underestimation of outage probability.

The findings are summarized in Tables II and III.

C. Discussion

The three models occupy distinct positions in the accuracy-versus-efficiency space. The Málaga model delivers the highest fidelity across turbulence regimes, but at a runtime cost exceeding $1700\times$ that of the lognormal surrogate. The Gamma-Gamma model occupies a practical middle ground, incurring only a $20\times$ cost increase over lognormal while retaining strong accuracy under moderate turbulence. The lognormal surrogate, obtained at negligible cost via MM parameter matching, is well suited to large-scale parameter sweeps and rapid link-budget screening. To illustrate the link-level consequence: the CCDF tail mismatch observed in Fig. 5 at $I = 0.1$ corresponds directly to an error in $P_{\text{out}} = P(I < 0.1)$

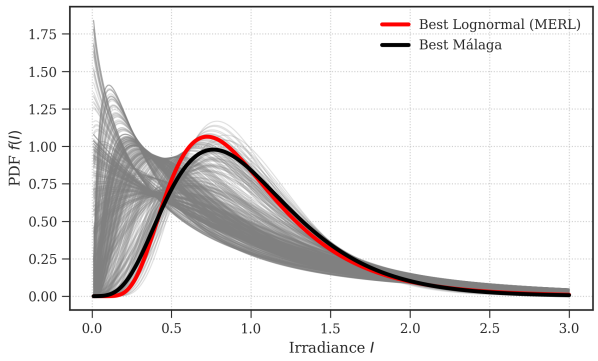

Fig. 4. PDF of the Málaga model and lognormal surrogate for representative parameter combinations. The Málaga PDF accommodates greater peak asymmetry and heavier tails than the Gamma-Gamma model, at low ρ and Ω.

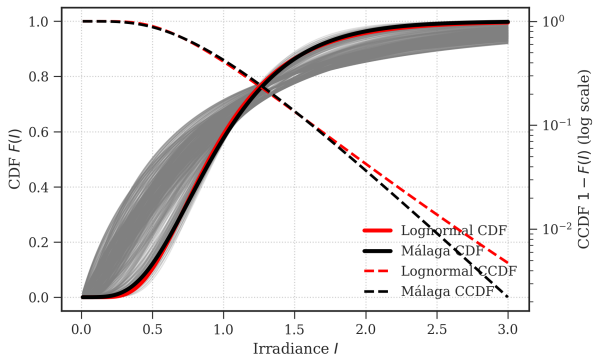

Fig. 5. CDF and CCDF of the Málaga model and lognormal surrogate. Tail mismatch in the CCDF is more severe than in the Gamma-Gamma case, confirming that the lognormal surrogate underestimates outage probability under strong or coherent-component-dominated turbulence.

of the same magnitude, which can amount to an order-of-magnitude underestimate of outage probability in coherence-dominated regimes. The cost difference between models is not merely numerical but structural, rooted in the mathematical complexity of each PDF.

Limitations: The present study isolates turbulence-induced irradiance fluctuations and does not explicitly incorporate pointing errors, aperture averaging, atmospheric attenuation, or measured experimental datasets. While these effects can influence practical FSO link performance, the objective here is to establish baseline fidelity and complexity relationships among widely used irradiance models. Extending the proposed framework to include additional propagation impairments and experimental validation represents an important direction for future work. Below we state a few remarks:

Remark 1 (Complexity Hierarchy): The lognormal surrogate requires only elementary arithmetic; the Gamma-Gamma model adds a single Bessel function evaluation $K_{\alpha-\beta}(\cdot)$ per irradiance point; and the Málaga model requires β evaluations of $K_{\alpha-k}(\cdot)$ together with the coefficient calculations in (7), so its per-point cost scales as $O(\beta)$ relative to Gamma-Gamma. The runtimes in Table III confirm this hierarchy empirically.

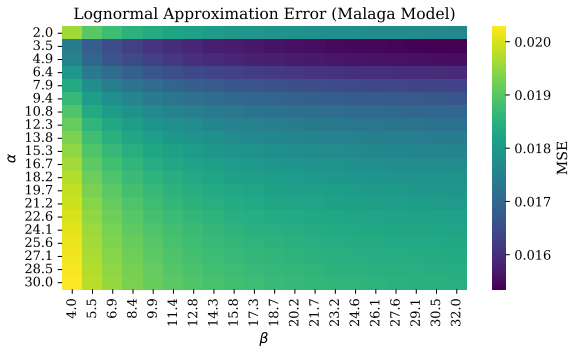

Fig. 6. Mean-squared error of the lognormal surrogate relative to the Málaga model over the explored parameter grid. Error is largest when ρ is small and Ω is large, corresponding to coherence-dominated propagation conditions.

TABLE II
QUALITATIVE SUMMARY OF MODEL CHARACTERISTICS

Model	Best Regime	Speed	Main Strength
Lognormal	Weak	High	Rapid MM surrogate
Gamma-Gamma	Moderate	Medium	Accuracy-speed balance
Málaga	Moderate-to-strong	Low	Maximum flexibility

From a system design perspective, the choice of model should be governed by the turbulence regime and the sensitivity of the target metric to tail behavior. For outage probability and BER prediction in moderate-to-strong turbulence, the tail accuracy of the Gamma-Gamma or Málaga model is necessary. For weak turbulence or early-stage feasibility analysis, the lognormal surrogate provides a computationally efficient and sufficiently accurate alternative.

An important observation is that statistical fidelity and computational cost do not increase proportionally. Although the Malaga model consistently achieves the highest accuracy, its computational cost is substantially greater than that of the Gamma-Gamma model and several orders of magnitude greater than that of the lognormal surrogate. Consequently, model selection should depend not only on turbulence severity but also on the scale of the intended analysis and the sensitivity of the target performance metric to lower-tail behavior.

Remark 2 (Model Selection): Model choice should be governed by the Rytov variance regime and the sensitivity of the target metric to tail behavior. The lognormal surrogate is appropriate for $\sigma_R^2 < 0.3$ and large-scale parameter screening; Gamma-Gamma is preferred for $0.3 \leq \sigma_R^2 \leq 1$; and Málaga should be used for $\sigma_R^2 > 1$ or whenever lower-tail accuracy is critical for outage or BER prediction.

Remark 3 (Hierarchical Strategy): A computationally efficient evaluation pipeline applies lognormal approximations for coarse screening across the full parameter space and invokes Gamma-Gamma or Málaga models selectively for configurations where tail fidelity is consequential, substantially reducing aggregate runtime without sacrificing accuracy.

TABLE III
MEAN PDF EVALUATION RUNTIME (2000 GRID POINTS, APPLE M1)

Model	Model (s)	Lognormal (s)	Speedup
Gamma-Gamma	0.0186	0.000937	19.9×
Málaga	1.180	0.000692	1705×

V. CONCLUSION

This paper presented a systematic fidelity-versus-computational-cost comparison of the Lognormal, Gamma-Gamma, and Malaga irradiance models for turbulence-induced fading in free-space optical channels. By combining MM surrogate construction, distribution-level error metrics, tail-behavior analysis, and runtime benchmarking, the study provides quantitative insight into the trade-offs associated with increasingly sophisticated turbulence models. The proposed lognormal surrogate enables rapid large-scale parameter exploration and motivates a hierarchical evaluation strategy in which high-fidelity models are reserved for scenarios where tail accuracy directly impacts performance predictions. Future work will investigate outage probability, BER, and experimentally validated channel models incorporating pointing errors, atmospheric attenuation, and aperture averaging.

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