Direction of arrival estimation using subspace based iterative orthogonal projection

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***Abstract*—The multiple signal classification is a widely adopted subspace method for direction of arrival estimation using phased sensor arrays. It estimates source direction(s) by exploiting the orthogonality between the signal and noise subspaces obtained from the data covariance matrix of the received signals. However, the method relies on eigenvalue decomposition and requires prior knowledge of the number of sources, which can limit its applicability in real scenarios. This paper proposes an iterative orthogonal projection method that preserves the subspace projection philosophy while avoiding explicit matrix decomposition. The proposal constructs the noise subspace iteratively by minimising a quadratic projection cost function and enforcing orthogonality among the resulting vectors. Additionally, a soft weighting strategy is applied to the projection vectors based on their associated energy measures, enabling implicit separation of signal and noise contributions without explicit model order knowledge. This results in a multiple signal classification look alike direction of arrival framework that is free of eigenvalue decomposition and agnostic to knowledge of model order. Simulation results demonstrate that the proposed method achieves equivalent performance to the classical subspace methods making it suitable for high noise and low snapshot regimes.**

***Index Terms*—Direction of arrival estimation, phased sensor arrays, multiple signal classification, iterative orthogonal projection, gradient descent.**

I. Introduction

Direction of arrival (DOA) estimation is a fundamental challenge in array signal processing including radar, sonar and wireless communications [1]. It involves determining the directions from which electromagnetic or acoustic signals, emitted by one or more sources within a surveillance region, impinge upon an array of spatially distributed sensors such as radar antennas, hydrophones, or microphones. At each time snapshot, each sensor records a superposition of noisy signals arriving from all active sources. Processing this ensemble of spatiotemporal measurements, using the relative phase and amplitude differences of the received signal wavefront across the sensor array, enables to infer the bearings (angles of arrival) of the sources, thereby enabling localisation and surveillance of source targets [2].

Beamforming, also known as delay-and-sum beamforming [3], [4], is the earliest technique for DOA estimation. The method suffers from limited resolution and poor performance in low signal-to-noise ratio (SNR) or with closely spaced sources. The Capon's minimum variance distortionless response (MVDR) beamformer [5] improves resolution by minimising output power while preserving gain in the desired direction but remains constrained by the Rayleigh limit. This constraint is overcome by the multiple signal classification (MUSIC) method [6]. Variants like Root MUSIC and ESPRIT [7], [8] enhance computational efficiency. Other recent DOA estimations can be found in [9]–[12]. The interest of this paper, is to formulate an alternative to MUSIC within the ambit of its underlying framework while overcoming its limitations.

The MUSIC method separates the signal and noise subspaces by eigenvalue decomposition of the sample data covariance matrix, and then evaluates a pseudospectrum by projecting candidate steering vectors onto the estimated noise subspace, producing sharp peaks at true DOAs. The eigenvalue decomposition of the sample covariance matrix is a computationally demanding matrix factorisation and must be recomputed whenever the covariance matrix changes. This makes it less suitable for large scale or real time applications where adaptive or incremental updates are required. Moreover, the algorithm assumes that the model order (a.k.a the number of sources) is known beforehand to dichotomise between the noise and signal subspaces. This assumption is apparently invalid in practical scenarios. This problem is overcome using model order selection techniques such as the Akaike information criterion (AIC) [13] and the minimum description length (MDL) [14]. The classical MUSIC is then applied using the estimated model order. Nevertheless, it is known that the AIC and MDL methods suffer from overestimation of the true model order under low SNR and limited snapshot conditions [15], [16], although the latter provides a more accurate estimate of the model order [16]. A recent MUSIC variant that does not require explicit knowledge of the number of sources was proposed in [17]. Unlike classical MUSIC, all the eigenvectors are used for candidate projection after being scaled with smoothly decaying weights such that the noise subspace is emphasised and the signal subspace is suppressed. However, performance of this method is limited under low

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SNR and limited snapshot conditions.

This paper develops an alternative subspace method that reproduces the philosophy of the MUSIC algorithm without suffering from the aforesaid limitations. Here the noise subspace is built iteratively by generating a set of orthonormal projection vectors, which serve as analogues of the noise eigenvectors in MUSIC. These vectors capture directions with minimum energy in the covariance matrix. Starting from random initial vectors, gradient based updates progressively steer each vector toward low-energy directions, while maintaining orthogonality among them. In this way, the vectors converge to an approximation of the noise subspace without requiring explicit eigenvalue decomposition. This framework is termed the iterative orthogonal projection (IOP). Furthermore, a soft weighting extension to the IOP framework is developed, and termed, soft IOP (SIOP), wherein the contribution of each projection vector is adaptively weighted based on its associated energy, thus avoiding any prior knowledge of the model order. Unlike iterative PCA [18], which estimates dominant variance directions, the proposed method constructs minimum energy directions corresponding to noise subspace analogues. The key contributions of this paper therefore are, (i) an eigen-free iterative scheme for approximating the noise subspace, and (ii) a non-reliance on precise model order estimation.

The rest of the paper is organised as follows. Section II describes the signal model. Conventional MUSIC and soft MUSIC [17] are described in section III. This is followed by the proposed method in section IV, simulation results in section V and conclusion in section VI.

II. The Signal Model

We consider DOA estimation of far-field sources, where incoming wavefronts are assumed to be planar due to the large source-to-array distance relative to the array aperture. Furthermore, the steering and signal modeling are performed under the narrowband assumption, which allows the use of frequency invariant array response vectors. Consider an array of M sensors receiving N narrowband signals from far-field sources. The received signal ensemble corresponding to a single snapshot is modeled as

$$\mathbf{X} = \mathbf{A}(\boldsymbol{\theta})\mathbf{S} + \mathbf{N} \tag{1}$$

where $\mathbf{X} = [\mathbf{x}(t=1), \cdots, \mathbf{x}(t=T)]$, $\mathbf{S} = [\mathbf{s}(t=1), \cdots, \mathbf{s}(t=T)]$ and $\mathbf{N} = [\mathbf{n}(t=1), \cdots, \mathbf{n}(t=T)]$ where T is the length of the ensemble. Here, $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_N)$ denotes the vector angles of arrival (in radians) of the N sources, the $\mathbf{A}(\boldsymbol{\theta})$ is the $M \times N$ steering matrix, $\mathbf{s}(t) \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N\times 1})$ is the source signal vector, and $\mathbf{n}(t) \sim \mathcal{CN}(\mathbf{0}, \sigma^2\mathbf{I}_{M\times 1})$ is additive white Gaussian noise with σ^2 denoting the noise variance corresponding to each of the M sensors. Note that $\mathcal{CN}(\mu, \Sigma)$ respectively denotes a complex Gaussian distribution with mean μ and covariance Σ, and $\mathbf{I}$ denotes identity matrix.

The steering matrix manifold is given by $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_N)]$. The nth column, called the steering vector, describes the phase shift across the sensors for a narrowband signal arriving from direction θ_n, and is described as

$$\mathbf{a}(\theta_n) = \exp\big(-j2\pi(d/\lambda)(m-1)\sin(\theta_n)\big), \tag{2}$$

for $m = 1, 2, \cdots, M$, where d is the spacing between sensors and λ is the wavelength.

The measurement in (1), which is usually demeaned over the length of the ensemble T to offset any constant bias ensuring that the covariance estimates reflect fluctuations due to the signal and noise, but not the mean, can be expressed as $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$. Then the sample data covariance matrix is given by

$$\widehat{\mathbf{R}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}(t)\mathbf{x}^H(t) = \frac{1}{T}\mathbf{X}\mathbf{X}^H. \tag{3}$$

III. Multiple Signal Classification

In this section, we first briefly describe the classical MUSIC algorithm as a precursor to the proposed method.

A. Classical MUSIC: Knowledge of model order is required

The MUSIC algorithm estimates DOAs by exploiting the orthogonality between the signal and noise subspaces of the sample covariance matrix. By performing eigendecomposition on $\widehat{\mathbf{R}}$, we obtain

$$\widehat{\mathbf{R}} = \mathbf{U}_s\boldsymbol{\Lambda}_s\mathbf{U}_s^H + \mathbf{U}_n\boldsymbol{\Lambda}_n\mathbf{U}_n^H \tag{4}$$

where $\mathbf{U}_s$ and $\mathbf{U}_n$ are the signal and noise subspace eigenvectors, respectively. The eigenvectors associate with the $M-N$ smallest eigenvalues together form the estimated noise subspace which is orthogonal to the signal subspace. The MUSIC value for a candidate angle θ is computed as

$$P_{\text{MUSIC}}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{U}_n\mathbf{U}_n^H\mathbf{a}(\theta)} \tag{5}$$

where $\mathbf{a}(\theta)$ is the steering vector for θ. Due to orthogonality of the noise and signal subspace, the denominator theoretically results in a 0 if the candidate θ lies in the signal subspace, and $P_{\text{MUSIC}}(\theta)$ will result in ∞. When computed over $\theta \in (-90°, 90°)$, the spectrum will result in sharp peaks for θ that correspond to the signal sources.

B. Soft MUSIC: Knowledge of model order is not required

Classical MUSIC requires the knowledge of the number of targets N to contruct the noise eigenvectors $\mathbf{U}_n$. Contrarily, the soft MUSIC [17] uses the full eigenstructure of the data covariance matrix. The eigenvectors $\mathbf{U} = [\mathbf{u}_1, \cdots, \mathbf{u}_M]$ are sorted in accordance to the eigenvalues such that $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_M)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$ and a weighting scheme

$$w_i = \exp\left(-\alpha \cdot \frac{\lambda_i - \min\big(\{\lambda_i\}_{i=1}^M\big)}{\max\big(\{\lambda_i\}_{i=1}^M\big) - \min\big(\{\lambda_i\}_{i=1}^M\big)}\right) \tag{6}$$

for $m = 1, \cdots, M$ is applied thereupon according to

$$\mathbf{U}_w = \mathbf{U} \cdot \text{diag}(\sqrt{w_1}, \cdots, \sqrt{w_M}) \tag{7}$$

such that eigenvectors lying in the noise subspace are weighted more than those lying in the signal subspace. Then the spectrum value, for a steering vector $\mathbf{a}(\theta)$ corresponding to a hypothesis angle θ is the inverse of its projection onto the full eigenstructure, and is computed according to

$$P_{\text{FULL}}(\theta) = \frac{1}{||\mathbf{U}_w^H \mathbf{a}(\theta)||^2} \tag{8}$$

This full eigenstructure method avoids explicit model order estimation by softly attenuating signal subspace contributions rather than excluding them. The method, however, is sensitive to the weight parameter α and the peak resolution may be poor due to minimal signal leakage into the noise subspace.

IV. Proposed iterative orthogonal projection

In this section, a subspace method that bypasses the eigenvalue decomposition iteratively constructing vectors that approximate the noise subspace is presented

A. IOP: Knowledge of model order is required

The noise directions are obtained by minimising the quadratic form

$$J(\mathbf{w}) = \mathbf{w}^H \widehat{\mathbf{R}} \mathbf{w} \tag{9}$$

subject to the constraint $\|\mathbf{w}\| = 1$ where $\mathbf{w} \in \mathbb{C}^M$ is a unit norm direction vector in the sensor space, initialised randomly. This quantity represents the energy of the received signal when projected onto the direction $\mathbf{w}$. The directions aligned with the signal subspace produce larger values of $J(\mathbf{w})$ and the directions dominated by noise produce smaller values. The minimisation is performed using a gradient descent [19] iteration

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \mu \widehat{\mathbf{R}} \mathbf{w}^{(k)} \tag{10}$$

where μ is a step size that controls the rate of convergence. A large step size can cause instability in the optimisation, while a small step size leads to slower convergence.

To ensure that multiple independent noise vectors are obtained, each newly computed vector is orthogonalised with respect to the previously obtained vectors. Let $\mathbf{n}_1, \ldots, \mathbf{n}_{n-1}$ denote previously estimated directions. The orthogonalisation step is given by

$$\mathbf{w} \leftarrow \mathbf{w} - \sum_{j=1}^{n-1} (\mathbf{n}_j^H \mathbf{w}) \mathbf{n}_j \tag{11}$$

The normalised vector $\mathbf{n}_n = \mathbf{w}/\|\mathbf{w}\|$ is then stored as the nth noise vector. Repeating this procedure yields the matrix

$$\mathbf{N}_{\text{IOP}} = [\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{d_n}] \tag{12}$$

which spans the estimated noise subspace, where the dimension of the noise subspace is $d_n = M - N$. The spatial spectrum can then be evaluated using the projection principle as

$$P_{\text{IOP}}(\theta) = \frac{1}{\|\mathbf{N}_{\text{IOP}}^H \mathbf{a}(\theta)\|^2} \tag{13}$$

For directions corresponding to actual sources, the steering vector $\mathbf{a}(\theta)$ becomes nearly orthogonal to the estimated noise subspace, causing the denominator to approach zero and producing a peak in the spatial spectrum.

The proposed IOP method preserves the subspace orthogonality principle while constructing the noise subspace iteratively, avoiding eigendecomposition. However, it still requires prior knowledge of model order to construct (12). The requirement is overcome in the subsequent section.

B. SIOP: Knowledge of model order is not required

In this strategy, we compute the projection vectors according to (12) for the full sensor space, i.e., M instead of N. Each projection vector $\mathbf{n}_k$ is then associated with a scalar energy

$$e_k = \mathbf{n}_k^H \widehat{\mathbf{R}} \mathbf{n}_k, \; k = 1, \cdots, M \tag{14}$$

This quantity corresponds to the Rayleigh quotient of $\widehat{\mathbf{R}}$ along the direction $\mathbf{n}_k$ and approximates the eigenvalue associated with that direction. Vectors aligned with the signal subspace produce larger energy values and those associated with noise directions produce smaller values.

We then define weights that softly emphasise directions towards noise and suppress directions towards the signal, according to

$$w_k = \exp\left(-\alpha \frac{e_k - e_{\min}}{e_{\max} - e_{\min}}\right) \tag{15}$$

for $k = 1, \cdots, M$ where α controls the sharpness of the weighting and $e_{\min}$ and $e_{\max}$ denote the minimum and maximum energy values. The weighted projection matrix is consequently defined as

$$\mathbf{W}_s = \mathbf{N}_{\text{IOP}} \, \text{diag}(\sqrt{w_1}, \ldots, \sqrt{w_M}) \tag{16}$$

The spatial spectrum is then computed as

$$P_{\text{SIOP}}(\theta) = \frac{1}{\|\mathbf{W}_s^H \mathbf{a}(\theta)\|^2} \tag{17}$$

Here, the vectors corresponding to smaller energy values receive larger weights effectively emphasising noise directions. Conversely, vectors aligned with the signal subspace are attenuated. Since the weighting is applied to all projection vectors, the proposed method does not require partitioning the space into signal and noise subspaces and therefore does not need explicit prior knowledge of the model order. The method also retains the computational advantage of avoiding matrix factorisation.

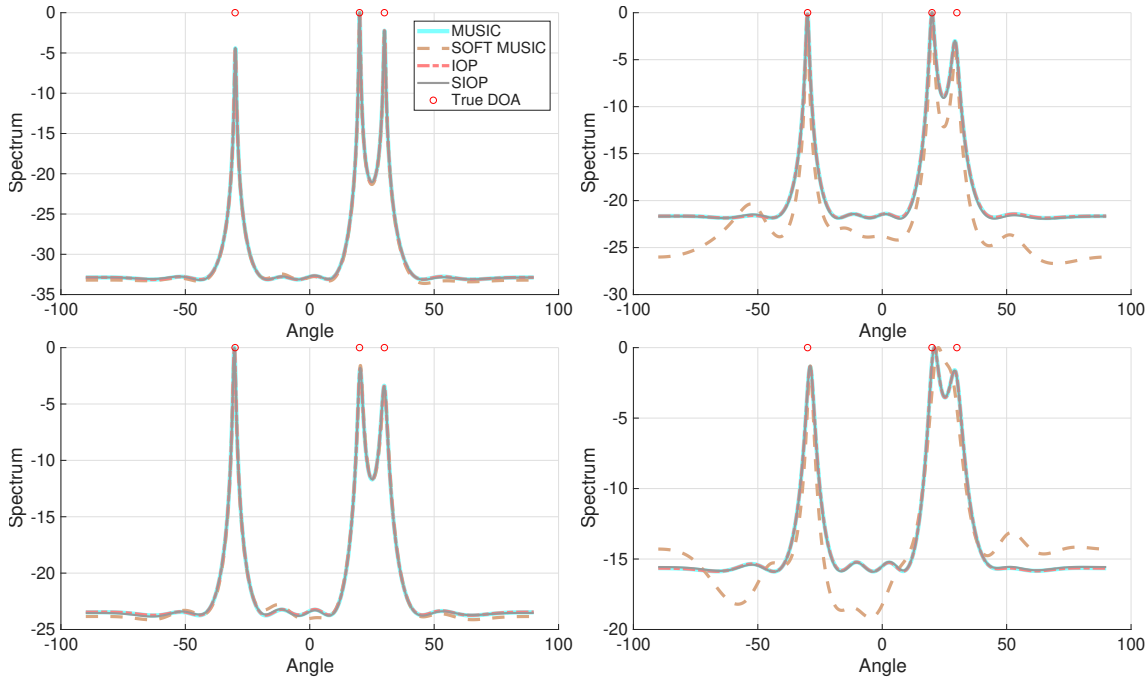

Fig. 1. The magnitude of the spectrum value versus the angle (in degrees). For the test case $(T, \text{SNR(dB)})$, the top left panel corresponds to $(100, 6)$, the top right to (100, -3), the bottom left to $(20, 3)$ and the bottom right to $(20, -3)$. The legend of the top left panel applies to the others also.

V. Simulation Results

In this section, we evaluate the performance of the proposed methods, the IOP and the SIOP, with the classical MUSIC [6] in which the model order is estimated by MDL [14] and the soft MUSIC [17]. We consider a uniform linear array consisting of $M = 10$ omnidirectional sensors equally spaced with an inter-element distance of $d = 0.5$ wavelengths and the operating wavelength is normalised to $\lambda = 1$. The results are averaged over 1000 Monte Carlo iterations. We assume the presence of three targets at $\theta = (\theta_1, \theta_2, \theta_3) = (-30°, 20°, 30°)$. The proposed method is implemented with a step size of $\mu = 0.001$ and 1000 iterations of gradient search with a weighting of $\alpha = 50$. These are chosen to ensure stable convergence of the iterative projection vectors. The small value of $\mu = 0.001$ prevents oscillatory updates of the gradient descent recursion while $\alpha = 50$ enables gradual attenuation of signal dominated directions without excessively suppressing intermediate components.

Four test scenarios, that include high noise and few snapshots, are examined as $(T, \text{SNR(dB)}) = (100, 6), (100, -3), (20, 6)$ and $(20, -3)$. Figure 1 shows the spectrum for the four test scenarios. It can be observed that at $T = 100$, all methods perform comparably to classical MUSIC. The more informative case is the high noise and snapshot-constrained scenario ($T = 20, \text{SNR} = -3\,\text{dB}$), where differences between methods become pronounced. Considering the MUSIC and IOP as the benchmarks, this study focusses on comparing SOFT MUSIC and the proposed SIOP as they do not assume knowledge of the model order. It is evident that soft MUSIC suffers from spurious fluctuations and the inability to consistently resolve closely spaced sources. This is because the soft weighting function over the full eigenstructure is unable to effectively separate the signal and noise subspaces, leading to leakage of signal components into the noise subspace. In contrast, the proposed SIOP method exhibits a spectrum that closely resembles the classical subspace methods with known model order. The peaks are well defined and aligned with the true DOAs, with minimal to no spurious responses. This is attributed to the accurate construction of projection vectors corresponding to the noise subspace through iterative gradient search.

For each of the four scenarios, we show in Table I the DOA estimation error which is the mean absolute difference between the true target and the estimated value, described as

$$\text{Er}(\theta_i) = \frac{1}{1000} \sum_{m=1}^{1000} |\widehat{\theta}_i^m - \theta_i| \tag{18}$$

and the mean absolute error in model order estimation described as spurious peak count error (SPCE) of the model order

$$\frac{1}{1000} \sum_{m=1}^{1000} |\#peaks(m) - N| \tag{19}$$

where $\#peaks$ is the number of peaks determined by $findpeaks(\cdot)$ function in MATLAB. It can be observed from

Method	SNR	Er(θ_1)	Er(θ_2)	Er(θ_3)	Er(θ_1)	Er(θ_2)	Er(θ_3)	SPCE	SPCE
		$T = 100$			$T = 20$			$T = 100$	$T = 20$
MUSIC	6 dB	0.0439	0.0646	0.0679	0.1084	0.1666	0.1801	5.0000	4.9850
	-3 dB	0.1480	0.2045	0.2176	0.3545	0.6137	0.6425	5.0000	4.9400
SOFT MUSIC	6 dB	0.0437	0.0650	0.0704	0.1112	0.1675	0.1842	4.8490	4.2960
	-3 dB	0.1809	0.2607	0.2801	0.4906	0.8812	0.9319	3.1530	3.0990
IOP	6 dB	0.0439	0.0646	0.0679	0.1086	0.1665	0.1793	5.0000	5.0000
	-3 dB	0.1480	0.2046	0.2176	0.3552	0.6074	0.6347	5.0000	4.9540
SIOP	6 dB	0.0434	0.0642	0.0674	0.1084	0.1668	0.1786	5.0000	5.0000
	-3 dB	0.1466	0.2035	0.2176	0.3545	0.5998	0.6380	5.0000	4.9340

TABLE I

THIS TABLE SHOWS THE DOA ESTIMATE ERROR $\widehat{\theta}$ FOR THE THREE SOURCES FROM (18) AND THE SPCE FROM (19) FOR $T = (100, 20)$ AND SNR$= (-3, 6)$ DECIBELS. TRUE DOA IS $\theta = (-30^\circ, 20^\circ, 30^\circ)$

the table that under favourable conditions $(T = 100, \text{SNR} = 6\,\text{dB})$, all methods exhibit comparable performance, with negligible differences in DOA estimation error. This is expected, as sufficient data and high SNR ensure clear separation between signal and noise subspaces. As the operating conditions become more challenging, i.e., at $(T = 100, \text{SNR} = -3\,\text{dB})$ and $(T = 20, \text{SNR} = -3\,\text{dB})$, clear distinctions emerge. The SOFT MUSIC exhibits noticeable degradation in estimation accuracy due to the reduced separation between signal and noise eigenvalues. This method fails to reliably distinguish between signal and noise contributions, leading to missed or merged peaks particularly when the sources are spatially close to one another. The proposed SIOP method, on the other hand, demonstrates superior robustness in these challenging scenarios. By constructing the projection space iteratively and employing energy based soft weighting, SIOP effectively emphasises noise dominated directions while suppressing signal components without requiring explicit model order estimation. This results in improved separation between signal and noise contributions even under limited data conditions. Consequently, SIOP achieves comparatively more accurate DOA estimates and consistently resolves all target peaks. The non-zero values of SPCE occur for all methods due to noise and finite-sample effects that introduce additional local maxima.

VI. CONCLUSION

This paper proposed an iterative orthogonal projection framework for DOA estimation as an alternative to classical MUSIC. By avoiding explicit eigenvalue decomposition and introducing a soft weighting mechanism, the method reduces dependence on model order knowledge and enhances robustness to noise and spatially close sources. Simulation results demonstrate that the proposed method achieves improved target resolution under limited snapshot and high noise conditions. In the future, the proposed method will be theoretically validated and applied to online adaptive settings and large array configurations.

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