E2 212: Matrix Theory

Topic: Vector Norms and Inner Product



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Vector Norms:

Vector Norm: Let V be a vector space over $\mathbb{F}$. A function $\|\cdot\| : V \to \mathbb{R}$ is a *norm* if $\forall\, \underline{x}, \underline{y} \in V$ and all $c \in \mathbb{F}$:

(1)	$\|\underline{x}\| \geq 0$	Non-negativity
(1a)	$\|\underline{x}\| = 0$ iff $\underline{x} = 0$	Positivity
(2)	$\|c\underline{x}\| = \|c\| \, \|\underline{x}\|$	Homogeneity
(3)	$\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$	Triangle inequality

Without (1a): the function is called a *semi-norm*.

Any non-zero vector $\underline{x}$ (in $\mathbb{R}^n$ or $\mathbb{C}^n$):

$$\underline{u} = \|\underline{x}\|^{-1}\underline{x} \quad \Longrightarrow \quad \|\underline{u}\| = \| \, \|\underline{x}\|^{-1}\underline{x} \, \| = \|\underline{x}\|^{-1}\|\underline{x}\| = 1.$$

These are *normed vector spaces*.

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\rho(\underline{x}) = |x_1|$, $\underline{y} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, $\rho(\underline{y}) = 0$
but $\underline{y} \neq 0$.

Inner Product: Let V be a vector space over the field F ($F = \mathbb{R}$ or $\mathbb{C}$). A function $\langle \cdot, \cdot \rangle : V \times V \to F$ is an *inner product* if for all $\underline{x}, \underline{y}, \underline{z} \in V$ and all $c \in \mathbb{F}$,

(1)	$\langle \underline{x}, \underline{x} \rangle \geq 0$	Non-negativity
(1a)	$\langle \underline{x}, \underline{x} \rangle = 0$ iff $\underline{x} = 0$	Positivity
(2)	$\langle \underline{x} + \underline{y}, \underline{z} \rangle = \langle \underline{x}, \underline{z} \rangle + \langle \underline{y}, \underline{z} \rangle$	Additivity
(3)	$\langle c\underline{x}, \underline{y} \rangle = c \, \langle \underline{x}, \underline{y} \rangle$	Homogeneity
(4)	$\langle \underline{x}, \underline{y} \rangle = \overline{\langle \underline{y}, \underline{x} \rangle}$	Hermitian property

Without (1a), $\langle \cdot, \cdot \rangle$ is a *semi-inner product*.

$\langle \underline{x}, \underline{y} \rangle = x_1 y_1$, $\underline{x} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, $\langle \underline{x}, \underline{x} \rangle = 0 \cdot 0 = 0$
with $\underline{x} \neq 0$.

The axioms (2), (3) and (4) say that $\langle \cdot, \cdot \rangle$ is a *sesquilinear* function.
The axioms (1a) and (1) require that $\langle \underline{x}, \underline{x} \rangle > 0$ if $\underline{x} \neq 0$.

$$\langle \underline{x}, \underline{y} \rangle = \|\underline{x}\|\|\underline{y}\| \cos\theta = \langle \underline{x}, \underline{y} \rangle = 0 \implies \underline{x} \perp \underline{y}.$$

Problem: Let $a, b, c, d \in \mathbb{F}$ and let $\underline{x}, \underline{y}, \underline{w}, \underline{z} \in \mathbb{F}^n$. Deduce the following properties of an inner product from the five axioms.

(a) $\langle \underline{x}, c\underline{y} \rangle = \bar{c} \langle \underline{x}, \underline{y} \rangle$

(b) $\langle \underline{x}, \underline{y} + \underline{z} \rangle = \langle \underline{x}, \underline{y} \rangle + \langle \underline{x}, \underline{z} \rangle$

(c) $\langle a\underline{x} + b\underline{y},\ c\underline{w} + d\underline{z} \rangle = a\bar{c} \langle \underline{x}, \underline{w} \rangle + b\bar{c} \langle \underline{y}, \underline{w} \rangle + a\bar{d} \langle \underline{x}, \underline{z} \rangle + b\bar{d} \langle \underline{y}, \underline{z} \rangle$

(d) $\langle \underline{x}, \langle \underline{x}, \underline{y} \rangle \underline{y} \rangle = \left| \langle \underline{x}, \underline{y} \rangle \right|^2$

(e) $\langle \underline{x}, \underline{y} \rangle = 0$ for all $\underline{y} \in V \iff \underline{x} = 0$

$\langle \underline{x}, \underline{y} \rangle = \underline{y}^H \underline{x} \Rightarrow \ \langle \underline{x}, c\underline{y} \rangle = (c\underline{y})^H \underline{x} = \bar{c}\, \underline{y}^H \underline{x} = \bar{c} \langle \underline{x}, \underline{y} \rangle$.

Theorem (Cauchy–Schwarz inequality)**.** Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space V over the field F ($F = \mathbb{R}$ or $\mathbb{C}$). Then

$$\left| \langle \underline{x}, \underline{y} \rangle \right|^2 \ \leq \ \langle \underline{x}, \underline{x} \rangle \langle \underline{y}, \underline{y} \rangle \qquad \text{for all } \underline{x}, \underline{y} \in V,$$

with equality iff $\underline{x}$ and $\underline{y}$ are linearly dependent, i.e. $\underline{x} = \alpha \underline{y}$ or $\underline{y} = \alpha \underline{x}$ for some $\alpha \in \mathbb{F}$.

Proof: Let $\underline{x}, \underline{y} \in V$ be given. If $\underline{x} = \underline{y} = 0$ there is nothing to prove, so we may assume that $\underline{y} \neq 0$. Let $\underline{u} = \langle \underline{y}, \underline{y} \rangle \underline{x} - \langle \underline{x}, \underline{y} \rangle \underline{y}$ and compute

$$\begin{aligned} 0 \leq \langle \underline{u}, \underline{u} \rangle &= \big\langle \langle \underline{y}, \underline{y} \rangle \underline{x} - \langle \underline{x}, \underline{y} \rangle \underline{y},\ \langle \underline{y}, \underline{y} \rangle \underline{x} - \langle \underline{x}, \underline{y} \rangle \underline{y} \big\rangle \\ &= \langle \underline{y}, \underline{y} \rangle^2 \langle \underline{x}, \underline{x} \rangle - \langle \underline{y}, \underline{y} \rangle \overline{\langle \underline{x}, \underline{y} \rangle} \langle \underline{x}, \underline{y} \rangle - \langle \underline{x}, \underline{y} \rangle \langle \underline{y}, \underline{x} \rangle \langle \underline{y}, \underline{y} \rangle + \langle \underline{y}, \underline{y} \rangle \overline{\langle \underline{x}, \underline{y} \rangle} \langle \underline{x}, \underline{y} \rangle \\ &= \langle \underline{y}, \underline{y} \rangle^2 \langle \underline{x}, \underline{x} \rangle - \langle \underline{y}, \underline{y} \rangle \left| \langle \underline{x}, \underline{y} \rangle \right|^2 \\ &= \langle \underline{y}, \underline{y} \rangle \Big(\langle \underline{x}, \underline{x} \rangle \langle \underline{y}, \underline{y} \rangle - \left| \langle \underline{x}, \underline{y} \rangle \right|^2 \Big) \end{aligned}$$

$$\implies \left(\langle \underline{x}, \underline{x}\rangle\langle \underline{y}, \underline{y}\rangle - \left|\langle \underline{x}, \underline{y}\rangle\right|^2\right) \geq 0 \qquad \left[\because\ \langle \underline{y}, \underline{y}\rangle > 0 \text{ by (3)}\right]$$

$$\implies \langle \underline{x}, \underline{x}\rangle\langle \underline{y}, \underline{y}\rangle \ \geq\ \left|\langle \underline{x}, \underline{y}\rangle\right|^2$$

$$\implies \left|\langle \underline{x}, \underline{y}\rangle\right|^2 \ \leq\ \langle \underline{x}, \underline{x}\rangle\langle \underline{y}, \underline{y}\rangle \qquad \textbf{proved.}$$

Corollary. If $\langle \cdot, \cdot \rangle$ is an inner product on a real or complex vector space V, then the function $\|\cdot\| : V \to [0, \infty]$ defined by $\|\underline{x}\|_2 = \langle \underline{x}, \underline{x}\rangle^{1/2}$ is a norm on V.

Non-negativity and positivity:

$$\|\underline{x}\|_2 = \langle \underline{x}, \underline{x}\rangle^{1/2}$$

(1a) $\langle \underline{x}, \underline{x}\rangle \geq 0$, equality holds iff $\underline{x} = 0$ (positivity)

(1) $\|\underline{x}\|_2 \geq 0$ non-negativity

(2) $\|\alpha\underline{x}\|_2 = \langle \alpha\underline{x}, \alpha\underline{x}\rangle^{1/2} = \langle \alpha, \alpha\rangle^{1/2}\langle \underline{x}, \underline{x}\rangle^{1/2} = |\alpha| \cdot \|\underline{x}\|_2$ (Homogeneity)

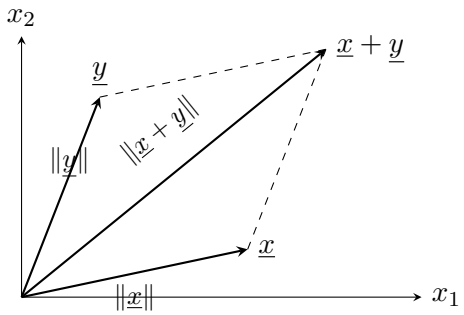
(3)
$$\begin{aligned}
\|\underline{x} + \underline{y}\|^2 &= \langle \underline{x} + \underline{y}, \underline{x} + \underline{y}\rangle \\
&= \langle \underline{x}, \underline{x}\rangle + \langle \underline{x}, \underline{y}\rangle + \overline{\langle \underline{x}, \underline{y}\rangle} + \langle \underline{y}, \underline{y}\rangle \\
&= \langle \underline{x}, \underline{x}\rangle + 2\,\mathrm{Re}\,\langle \underline{x}, \underline{y}\rangle + \langle \underline{y}, \underline{y}\rangle \\
&\leq \langle \underline{x}, \underline{x}\rangle + \langle \underline{y}, \underline{y}\rangle + 2\left|\langle \underline{x}, \underline{y}\rangle\right| \\
&\leq \|\underline{x}\|^2 + \|\underline{y}\|^2 + 2\,\langle \underline{x}, \underline{x}\rangle^{1/2}\langle \underline{y}, \underline{y}\rangle^{1/2} \\
&= \left(\|\underline{x}\| + \|\underline{y}\|\right)^2
\end{aligned}$$

$$2\,\mathrm{Re}\,\langle \underline{x}, \underline{y}\rangle \leq 2\left|\langle \underline{x}, \underline{y}\rangle\right| \leq 2\,\langle \underline{x}, \underline{x}\rangle^{1/2}\langle \underline{y}, \underline{y}\rangle^{1/2}$$

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Norm:

$$\|\cdot\| : V \to \mathbb{R}, \qquad \|\underline{x}+\underline{y}\| \le \|\underline{x}\| + \|\underline{y}\|$$



Parallelogram identity:

$$\tfrac{1}{2}\Big(\|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2\Big) = \|\underline{x}\|^2 + \|\underline{y}\|^2$$

To check if a norm is induced from the inner product, the above identity is required.

Backward triangle inequality:

$$\big|\ \|\underline{x}\| - \|\underline{y}\|\ \big| \ \le\ \|\underline{x}-\underline{y}\| \qquad \equiv \qquad \pm\big(\|\underline{x}\| - \|\underline{y}\|\big) \le \|\underline{x}-\underline{y}\|$$

$$\begin{aligned} \|\underline{x}\| &= \|\underline{x}-\underline{y}+\underline{y}\| \\ &\le \|\underline{x}-\underline{y}\| + \|\underline{y}\| \\ \implies \quad \|\underline{x}\| - \|\underline{y}\| &\le \|\underline{x}-\underline{y}\| \end{aligned} \tag{1}$$

Similarly:

$$\begin{aligned} \|\underline{y}\| &= \|\underline{x}-\underline{y}-\underline{x}\| \\ &\le \|\underline{x}-\underline{y}\| + \|\underline{x}\| \\ \implies \quad \|\underline{y}\| - \|\underline{x}\| &\le \|\underline{x}-\underline{y}\| \end{aligned} \tag{2}$$

(1) & (2) $\implies \pm\big(\|\underline{x}\| - \|\underline{y}\|\big) \le \|\underline{x}-\underline{y}\|$.

Proof of parallelogram theorem:

$$\|\underline{x}\| = \langle \underline{x}, \underline{x} \rangle^{1/2} \qquad \therefore \ \|\underline{x} + \underline{y}\| = \langle \underline{x} + \underline{y}, \underline{x} + \underline{y} \rangle^{1/2}$$

$$\begin{aligned} \|\underline{x} + \underline{y}\|^2 &= \langle \underline{x} + \underline{y}, \underline{x} + \underline{y} \rangle \\ &= \langle \underline{x}, \underline{x} \rangle + \langle \underline{x}, \underline{y} \rangle + \langle \underline{y}, \underline{x} \rangle + \langle \underline{y}, \underline{y} \rangle \\ &= \|\underline{x}\|^2 + \|\underline{y}\|^2 + 2\,\mathrm{Re}\big\{\langle \underline{x}, \underline{y} \rangle\big\} \end{aligned} \tag{1}$$

Similarly

$$\|\underline{x} - \underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2 - 2\,\mathrm{Re}\big\{\langle \underline{x}, \underline{y} \rangle\big\} \tag{2}$$

(1) & (2) gives

$$\|\underline{x} + \underline{y}\|^2 + \|\underline{x} - \underline{y}\|^2 = 2\big(\|\underline{x}\|^2 + \|\underline{y}\|^2\big)$$

$$\implies \quad \tfrac{1}{2}\Big[\|\underline{x} + \underline{y}\|^2 + \|\underline{x} - \underline{y}\|^2\Big] = \|\underline{x}\|^2 + \|\underline{y}\|^2 \qquad \textbf{proved.}$$

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(1) The **Euclidean norm** (ℓ_2-norm) of a vector $\underline{x} = [x_1 \ \cdots \ x_n]^T \in \mathbb{C}^n$:

$$\|\underline{x}\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2}$$

This norm helps to measure the length of a vector from the origin.

(2) The **sum norm** (ℓ_1-norm) on $\mathbb{C}^n$ is:

$$\|\underline{x}\|_1 = |x_1| + \cdots + |x_n|$$

(3) The **max-norm** (or ℓ_∞-norm) on $\mathbb{C}^n$ is:

$$\|\underline{x}\|_\infty = \max\{|x_1|, \ \ldots, \ |x_n|\}$$

(4) The **ℓ_p-norm** on $\mathbb{C}^n$ is:

$$\|\underline{x}\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}, \qquad p \geq 1$$

Only the ℓ_2-norm follows the parallelogram identity.

(5) ℓ_0-norm (pseudo norm):

$$\|\underline{x}\|_0 = \sum_{i=1}^{n} \mathbb{1}\{x_i \neq 0\} = \text{non-zero}(\underline{x})$$

It is *not* a norm.

Ex. $x_1 = 1$ and $x_i = 0$, $i \neq 1$ (given), $\alpha = 2$.

$$\|\underline{x}\|_0 = 1, \qquad \|\alpha\underline{x}\|_0 = 1 \neq 2 = |\alpha| \, \|\underline{x}\|_0,$$

hence not a norm — it does not satisfy homogeneity.

⋆ **ℓ_p-norm for $0 \leq p < 1$ is not a norm:**

$$\left.\begin{aligned} x_1 = 2, \quad x_i = 0 \ \ \forall\, i \neq 1 \\ y_2 = 2, \quad y_i = 0 \ \ \forall\, i \neq 2 \end{aligned}\right\} \qquad \begin{aligned} \|\underline{x}\|_p + \|\underline{y}\|_p &= 2 + 2 = 4 \\ \|\underline{x} + \underline{y}\|_p &= \left[2^p + 2^p\right]^{1/p} = 2 \cdot 2^{1/p} > 4 \end{aligned}$$

∘ **All equivalent terms:**

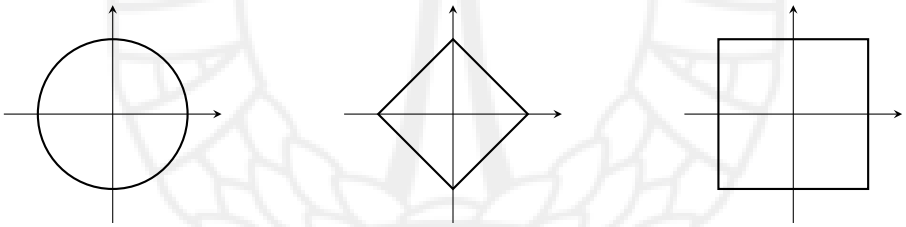
$$\|\underline{x}\|_\infty = \lim_{p\to\infty} \|\underline{x}\|_p = \lim_{p\to\infty} \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} = \max_i |x_i|.$$

Proof: Let $\underline{\tilde{x}} = [\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k, \tilde{x}_{k+1}, \ldots, \tilde{x}_n]^T$ with

$$\tilde{x}_1 = \max_i |x_i| \quad \text{for } i = 1 \text{ to } k$$

$\Longrightarrow$ for $\tilde{x}_{k+1}, \ldots, \tilde{x}_n$ we have $\left|\dfrac{\tilde{x}_i}{\tilde{x}_1}\right| < 1$ for $i = k+1, \ldots, n$.

$$\begin{aligned} \|\underline{x}\|_p &= |\tilde{x}_1| \left(k + \left|\tfrac{\tilde{x}_{k+1}}{\tilde{x}_1}\right|^p + \cdots + \left|\tfrac{\tilde{x}_n}{\tilde{x}_1}\right|^p\right)^{1/p} \\ &= |\tilde{x}_1| = \max_i |x_i|. \end{aligned}$$



ℓ_2-norm $\quad$ ℓ_1-norm $\quad$ ℓ_∞-norm

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Equivalent Norm: Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be given norms on a finite dimensional real or complex vector space. Then

$$c_m\|\underline{x}\|_b \;\le\; \|\underline{x}\|_a \;\le\; C_M\|\underline{x}\|_b$$
$$\|\underline{x}\|_2 \;\le\; \|\underline{x}\|_1 \;\le\; \sqrt{n}\,\|\underline{x}\|_2$$

Lemma. Every vector norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ is a continuous function: for every $\epsilon > 0$ there corresponds $\delta > 0$ such that

$$\big|\;\|\underline{x}\| - \|\underline{y}\|\;\big| < \epsilon \qquad \text{whenever} \qquad |x_i - y_i| < \delta.$$

(*proof: next page*)

Problem: For each pair of norms $\|\cdot\|_a, \|\cdot\|_b$ on an n-dimensional space V, exhibit positive constants α and β (depending only on the norms) such that

$$\alpha \;\le\; \frac{\|\underline{x}\|_a}{\|\underline{x}\|_b} \;\le\; \beta \qquad \text{for all non-zero vectors in } V. \tag{1}$$

Sol^n: For $S_b = \{\,\underline{y} \;:\; \|\underline{y}\|_b = 1\,\}$, let $\mu = \min\limits_{\underline{y}\in S_b}\|\underline{y}\|_a > 0$, and write $\dfrac{\underline{x}}{\|\underline{x}\|_b} \in S_b$

$$\Longrightarrow \quad \|\underline{x}\|_a = \|\underline{x}\|_b \left\|\frac{\underline{x}}{\|\underline{x}\|_b}\right\|_a \;\ge\; \|\underline{x}\|_b \min_{\underline{y}\in S_b}\|\underline{y}\|_a = \mu\,\|\underline{x}\|_b$$

The same argument shows there is a $\upsilon > 0$ such that $\|\underline{x}\|_b \ge \upsilon\,\|\underline{x}\|_a$. Thus (1) gives $\alpha = \mu$ and $\beta = 1/\upsilon$.

Similarly (1) ensures that

$$\|\underline{x}_k - \underline{x}\|_a \to 0 \quad \text{iff} \quad \|\underline{x}_k - \underline{x}\|_b \to 0.$$

Let $\upsilon = \max\limits_{\underline{y}\in S_b}\|\underline{y}\|_a > 0$,

$$\|\underline{x}\|_a = \|\underline{x}\|_b \left\|\frac{\underline{x}}{\|\underline{x}\|_b}\right\|_a \;\le\; \|\underline{x}\|_b \max_{\underline{y}\in S_b}\|\underline{y}\|_a = \upsilon\,\|\underline{x}\|_b.$$

Proof: e_i = canonical vector, $e_1 = \begin{bmatrix}1\\0\\0\\\vdots\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\\\vdots\end{bmatrix}$

$$\begin{aligned}\|\underline{x}-\underline{y}\| &= \Big\|\sum_{i=1}^{n}(x_i-y_i)\,e_i\Big\| \\ &\le \sum_{i=1}^{n}\big\|(x_i-y_i)\,e_i\big\| \\ &= \sum_{i=1}^{n}|x_i-y_i|\,\|e_i\| \\ &\le \max_i\|e_i\|\sum_{i=1}^{n}|x_i-y_i| \\ &\le n\,\upsilon\,\delta \qquad\qquad \text{for } |x_i-y_i|<\delta \\ \implies \quad \|\underline{x}-\underline{y}\| &\le n\,\upsilon\,\delta\end{aligned}$$

From the backward triangular inequality,

$$\big|\;\|\underline{x}\|-\|\underline{y}\|\;\big| \;\le\; \|\underline{x}-\underline{y}\| \;\le\; n\,\upsilon\,\delta=\epsilon.$$

◦ **K-norm:**

$$\|\underline{x}\|_{[k]} = |x_{i_1}|+|x_{i_2}|+\cdots+|x_{i_k}| \quad \text{in which } |x_{i_1}|\ge|x_{i_2}|\ge\cdots\ge|x_{i_n}|.$$

$$\underline{x}=(3,\,-1,\,2i,\,-4i)\in\mathbb{C}^4$$

$$|x_1|=3,\quad |x_2|=1,\quad |x_3|=2,\quad |x_4|=4 \qquad\Longrightarrow\qquad 4>3>2>1$$

$$\|\underline{x}\|_{[1]}=4,\qquad \|\underline{x}\|_{[2]}=4+3=7,\qquad \|\underline{x}\|_{[3]}=4+3+2=9.$$

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⋆ A sequence $\{\underline{x}_k\}$ of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ converges to $\underline{x}$ w.r.t. $\|\cdot\|$:

$$\lim_{k\to\infty} \|\underline{x}_k - \underline{x}\| = 0 \qquad \text{or} \qquad \lim_{k\to\infty} \underline{x}_k = \underline{x} \quad \text{w.r.t. } \|\cdot\|.$$

⋆ If $\|\cdot\|_a$ and $\|\cdot\|_b$ are two *equivalent* norms, then $\lim_{k\to\infty} \underline{x}_k = \underline{x}$ w.r.t. $\|\cdot\|_a$ iff $\lim_{k\to\infty} \underline{x}_k = \underline{x}$ w.r.t. $\|\cdot\|_b$.

Pre-norm: on a finite dim. $\mathbb{R}/\mathbb{C}$ vector space. A real-valued function that satisfies positivity, non-negativity, homogeneity and continuity (without sub-additivity) is called a *pre-norm*.

Dual-norm: Let $f(\cdot)$ be a pre-norm on $V = \mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). The function

$$f^D(\underline{y}) = \max_{f(\underline{x})=1} \text{Re}\,\langle \underline{x}, \underline{y}\rangle = \max_{f(\underline{x})=1} \text{Re}\, \underline{y}^H\underline{x} = \max_{\underline{x}\neq 0} \frac{|\underline{y}^H\underline{x}|}{f(\underline{x})}$$

is the *dual norm* of f.

Proof:

$$|\underline{y}^H\underline{x}| = \text{Re}\big(c\,\underline{y}^H\underline{x}\big) \quad \text{for } |c|=1 \qquad = \big|c\,\underline{y}^H\underline{x}\big| \quad \text{for } |c|=1$$

Proof:

$$\begin{aligned}
\max_{f(\underline{x})=1} |\underline{y}^H\underline{x}| &= \max_{f(\underline{x})=1} \max_{|c|=1} \text{Re}\big(c\,\underline{y}^H\underline{x}\big) \\
&= \max_{f(\underline{x})=1} \max_{|c|=1} \text{Re}\big(\underline{y}^H(c\underline{x})\big) \\
&= \max_{|c|=1} \max_{f(\underline{x}/c)=1} \text{Re}\big(\underline{y}^H\underline{x}\big) \\
&= \max_{|c|=1} \max_{f(\underline{x})=1} \text{Re}\big(\underline{y}^H\underline{x}\big) \\
&= \max_{f(\underline{x})=1} \text{Re}\big(\underline{y}^H\underline{x}\big) = \max_{\underline{x}\neq 0} \frac{|\underline{y}^H\underline{x}|}{f(\underline{x})}.
\end{aligned}$$

⋆

$$f^D(\underline{y}) = \|\underline{y}\|_* = \max_{\|\underline{x}\|=1} |\underline{y}^*\underline{x}| = \max \frac{|\underline{y}^*\underline{x}|}{\|\underline{x}\|}$$

$$\underline{y}^H\underline{x} = \underline{y}^*\underline{x} = \begin{cases} \underline{y}^T\underline{x}, & \mathbb{R}^n \\ \bar{\underline{y}}^T\underline{x}, & \mathbb{C}^n \end{cases} \qquad \text{is a linear function.}$$

Observe that the dual norm is a well-defined function on V because $\mathrm{Re}\,[\underline{y}^H\underline{x}]$ is a continuous function of $\underline{x}$ for each fixed $\underline{y} \in V$ and the set $\{\,\underline{x} : f(\underline{x}) = 1\,\}$ is compact. If c is a scalar such that $|c| = 1$, then the homogeneity of f permits us to compute.

$\star$ The function $f^D(\cdot)$ is evidently homogeneous; it is positive for $\underline{y} \neq 0$: homogeneity of $f(\cdot)$ ensures that

$$f^D(\underline{y}) = \max_{f(\underline{x})=1} |\underline{y}^H\underline{x}| \;\geq\; \left|\underline{y}^H \frac{\underline{y}}{f(\underline{y})}\right| = \frac{\|\underline{y}\|_2^2}{f(\underline{y})} > 0 \qquad \textcircled{1}\ \text{positive}$$

Triangle inequality: ③ $f(\cdot)$ does not obey the triangle inequality, $f^D(\cdot)$ always does.

Proof:

$$\begin{aligned} f^D(\underline{y}+\underline{z}) &= \max_{f(\underline{x})=1} \left|(\underline{y}+\underline{z})^H\underline{x}\right| \;\leq\; \max_{f(\underline{x})=1} \left[|\underline{y}^H\underline{x}| + |\underline{z}^H\underline{x}|\right] \\ &= \max_{f(\underline{x})=1} |\underline{y}^H\underline{x}| \;+\; \max_{f(\underline{x})=1} |\underline{z}^H\underline{x}| \\ &= f^D(\underline{y}) + f^D(\underline{z}) \\ \Longrightarrow \quad f^D(\underline{y}+\underline{z}) &\leq f^D(\underline{y}) + f^D(\underline{z}) \end{aligned}$$

① The dual norm of a pre-norm is positive.

② The dual norm of a pre-norm is homogeneous.

Proof:

$$f^D(c\underline{y}) = \max_{f(\underline{x})=1} \left|c\,\underline{y}^H\underline{x}\right| = |c| \max_{f(\underline{x})=1} \left|\underline{y}^H\underline{x}\right| = |c|\, f^D(\underline{y}).$$

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Generalization of C–S: Let $f(\cdot)$ be a pre-norm, then $\forall\, \underline{x}, \underline{y} \in V$

$$|\underline{y}^H \underline{x}| \leq f(\underline{x})\, f^D(\underline{y}) \tag{1}$$

$$|\underline{y}^H \underline{x}| \leq f^D(\underline{x})\, f(\underline{y}) \tag{2}$$

Proof: If $\underline{x} \neq 0$,

$$\frac{|\underline{y}^H \underline{x}|}{f(\underline{x})} \leq \max_{f(\underline{z})=1} |\underline{y}^H \underline{z}| = f^D(\underline{y})$$

$$\implies \quad \frac{|\underline{y}^H \underline{x}|}{f(\underline{x})} \leq f^D(\underline{y})$$

$$\implies \quad |\underline{y}^H \underline{x}| \leq f^D(\underline{y})\, f(\underline{x})$$

Similarly $|\underline{x}^H \underline{y}| \leq f^D(\underline{x})\, f(\underline{y})$

$$\implies \quad |\underline{x}^H \underline{y}| = |\underline{y}^H \underline{x}| \leq f^D(\underline{x})\, f(\underline{y}).$$

Let $f(\cdot)$ and $g(\cdot)$ be pre-norms on $\mathbb{R}^n$ or $\mathbb{C}^n$ and let $c > 0$ be given. Then,

(a) $cf(\cdot)$ is a pre-norm with its dual norm $c^{-1} f^D(\cdot)$.

(b) $f(\underline{x}) \leq g(\underline{x}) \;\; \forall\, \underline{x} \in \mathbb{R}^n$ or $\mathbb{C}^n$, then

$$f^D(\underline{y}) \;\geq\; g^D(\underline{y})$$

(c) $f^{DD}(\underline{x}) \leq f(\underline{x})$

(d) $f^{DD}(\underline{y}) \leq f(\underline{y})$

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Proof: (a) If $f(\cdot)$ is continuous then $cf(\cdot)$ is continuous for $c > 0$.

$$\implies \quad f_c^D(\underline{y}) = \max_{\underline{x}\neq 0} \left| \frac{\underline{y}^H \underline{x}}{cf(\underline{x})} \right| = \max_{\underline{x}\neq 0} \frac{|\underline{y}^H \underline{x}|}{c\, f(\underline{x})} = c^{-1} f^D(\underline{y}).$$

(b)

$$f^D(\underline{y}) = \max_{\underline{x}\neq 0} \frac{|\underline{y}^H \underline{x}|}{f(\underline{x})} \;\geq\; \max_{\underline{x}\neq 0} \frac{|\underline{y}^H \underline{x}|}{g(\underline{x})} = g^D(\underline{y}) \qquad \text{as } f(\underline{x}) \leq g(\underline{x}).$$

(c)

$$\begin{aligned} f^{DD}(\underline{x}) &= \max_{f^D(\underline{y})=1} |\underline{y}^H \underline{x}| \\ &\leq \max_{f^D(\underline{y})=1} f(\underline{x})\, f^D(\underline{y}) \\ &= f(\underline{x}) \\ \implies \quad f^{DD}(\underline{x}) &\leq f(\underline{x}) \end{aligned}$$

(d) $\star$ If $\underline{x}, \underline{y} \in \mathbb{F}^n$, then

$$|\underline{y}^H \underline{x}| \leq \|\underline{y}\|_\infty \|\underline{x}\|_1 \qquad \text{and} \qquad |\underline{y}^H \underline{x}| \leq \|\underline{x}\|_\infty \|\underline{y}\|_1$$

Proof:

$$\begin{aligned} |\underline{y}^H \underline{x}| = \Big|\sum_{i=1}^n \bar{y}_i x_i\Big| \;&\leq\; \sum_{i=1}^n \big|\bar{y}_i x_i\big| \\ &\leq \sum_{i=1}^n |\bar{y}_i|\, |x_i| \;\leq\; \max_{1\leq i\leq n} |y_i| \cdot \sum_{j=1}^n |x_j| \;=\; \|\underline{y}\|_\infty \|\underline{x}\|_1 \\ \implies \quad |\underline{y}^H \underline{x}| &\leq \|\underline{y}\|_\infty \|\underline{x}\|_1, \qquad \text{similarly } |\underline{y}^H \underline{x}| \leq \|\underline{x}\|_\infty \|\underline{y}\|_1. \end{aligned}$$

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⋆ **Dual norm of $\|\cdot\|_2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$**

For the ℓ_2-norm: the dual norm is again the ℓ_2-norm

$$f^D(\underline{y}) = \left| \underline{y}^H \cdot \frac{\underline{y}}{\|\underline{y}\|_2} \right| = \|\underline{y}\|_2.$$

$$\|\underline{y}\|_1^D = \max_{\|\underline{x}\|_1=1} |\underline{y}^H \underline{x}| \;\le\; \max_{\|\underline{x}\|_1=1} \|\underline{y}\|_\infty \|\underline{x}\|_1 \;\le\; \|\underline{y}\|_\infty$$

$$\|\underline{y}\|_\infty^D = \max_{\|\underline{x}\|_\infty=1} |\underline{y}^H \underline{x}| = \max_{\|\underline{x}\|_\infty=1} \|\underline{x}\|_\infty \|\underline{y}\|_1 = \|\underline{y}\|_1$$

Let $|y_k| = \|\underline{y}\|_\infty$, $x_k = \dfrac{y_k}{|y_k|}$ for $y_k \neq 0$, $x_i = 0$ $(i \neq k)$.
Then $\|\underline{x}\|_1 = 1$, $\underline{y}^H \underline{x} = \bar{y}_k \dfrac{y_k}{|y_k|} = \|\underline{y}\|_\infty$.

Hölder's inequality:

$$|\underline{y}^H \underline{x}| \;\le\; \|\underline{x}\|_p \, \|\underline{y}\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad p \ge 1 \text{ and } q \ge 1$$

$$\|\cdot\|_p^D = \|\cdot\|_q \qquad \text{and} \qquad \|\cdot\|_q^D = \|\cdot\|_p.$$

Theorem. Let $\|\cdot\|$ be a norm on $V = \mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and let $c > 0$ be given. Then $\|\underline{x}\| = c\,\|\underline{x}\|^D$ $\forall\, \underline{x} \in V$ iff

$$\textcircled{1} \qquad \|\cdot\| = \sqrt{c}\,\|\cdot\|_2,$$

in particular $\|\cdot\| = \|\cdot\|^D$ iff $\|\cdot\| = \|\cdot\|_2$.

Proof: Assume $\|\cdot\| = \sqrt{c}\,\|\cdot\|_2$

$$\|\cdot\|^D = \frac{1}{\sqrt{c}}\|\cdot\|_2^D = \frac{1}{\sqrt{c}}\|\cdot\|_2 = \frac{1}{\sqrt{c}} \cdot \frac{1}{\sqrt{c}}\|\cdot\| = \frac{1}{c}\|\cdot\|$$

Conversely, $N(\underline{x}) = c^{-1/2}\|\underline{x}\|$ be a norm. So $\|\underline{x}\| = c\,\|\underline{x}\|^D$ ensures

$$N^D(\cdot) = c^{1/2}\|\cdot\|^D = c^{1/2}c^{-1}\|\cdot\| = c^{-1/2}\|\cdot\|$$

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So $N(\cdot)$ is *self-dual*.

$$\|\underline{x}\|_2^2 = |\underline{x}^H \underline{x}| \;\leq\; N(\underline{x})\, N^D(\underline{x}) \qquad \forall\, \underline{x} \in V$$
$$\Longrightarrow \quad \|\underline{x}\|_2 \leq N(\underline{x})$$

But $f(\underline{x}) \geq g(\underline{x}) \Rightarrow f^D(\underline{x}) \leq g^D(\underline{x})$

$$\|\underline{x}\|_2 \;\geq\; N(\underline{x}) \qquad \Longrightarrow \qquad N(\cdot) = \|\cdot\|_2.$$

Extra notes:

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