E2 212: Matrix Theory

Topic: Eigenvalues and Eigenvectors



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Eigenvalues and Eigenvectors.

① Quadratic program with quadratic equality constraint

$$\underset{\underline{x}\in\mathbb{R}^n}{\text{maximize}} \;\; \underline{x}^T A\underline{x} \qquad \text{s.t.} \quad \underline{x}^T\underline{x} = 1$$

We solve by Lagrange

$$\mathcal{L}(\underline{x},\lambda) = \underline{x}^T A\underline{x} - \lambda(\underline{x}^T\underline{x} - 1)$$

$$d\mathcal{L} = 0 \implies 2(A\underline{x} - \lambda\underline{x}) = 0 \implies A\underline{x} = \lambda\underline{x}$$

$$\begin{aligned} d &= \underline{x}^T\underline{x} \\ dd &= d\underline{x}^T\,\underline{x} + \underline{x}^T d\underline{x} \\ &= (d\underline{x})^T\underline{x} + \underline{x}^T d\underline{x} \\ &= \underline{x}^T d\underline{x} + \underline{x}^T d\underline{x} \\ &= 2\,\underline{x}^T d\underline{x} \end{aligned}$$

$$\begin{aligned} f(\underline{x}) &= \underline{x}^T A\underline{x} \quad [A = I \text{ gives the box on the left}] \\ df &= (d\underline{x})^T A\underline{x} + \underline{x}^T A\,(d\underline{x}) \\ &= \underline{x}^T A^T d\underline{x} + \underline{x}^T A\,d\underline{x} \\ &= \underline{x}^T(A^T + A)\,d\underline{x} \\ &= 2\,\underline{x}^T A\,d\underline{x} \quad (A = A^T) \\ \frac{df}{d\underline{x}} &= 2A\underline{x} \end{aligned}$$

② Let say $A_{n\times n}$ diagonalizable

$$A = V\Lambda V^{-1}$$

where $V = [\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n]$, & $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

$$\begin{aligned} &\implies AV = V\Lambda \\ &\implies A[\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n] = [\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n]\,\Lambda \\ &\implies [A\underline{v}_1,\; A\underline{v}_2,\; \ldots,\; A\underline{v}_n] = [\lambda_1\underline{v}_1,\; \lambda_2\underline{v}_2,\; \ldots,\; \lambda_n\underline{v}_n] \\ &\implies A\underline{v}_i = \lambda_i\underline{v}_i \implies A\underline{x} = \lambda\underline{x} \\ &\implies (A - \lambda I)\underline{x} = 0 \implies \det(A - \lambda I) = 0 \end{aligned}$$

Theorem: Let $A \in M_n$ be given. Then A has an eigenvalue. In fact, for each given nonzero vector $\underline{y} \in \mathbb{C}^n$, there is a polynomial $g(t)$ of degree at most $n-1$ such that $g(A)\underline{y}$ is an eigenvector of A.

Proof: Let say $\underline{y}, A\underline{y}, A^2\underline{y}, \ldots, A^k\underline{y}$ are L.D. for least integer $m \leq k$

$$\implies \text{for} \quad a_m A^m \underline{y} + a_{m-1} A^{m-1}\underline{y} + \cdots + a_1 A\underline{y} + a_0 \underline{y} = 0, \qquad a_0, a_1, \ldots, a_m \text{ not all } 0.$$

For $a_m = 0$, $\underline{y}, A\underline{y}, A^2\underline{y}, \ldots, A^{m-1}\underline{y}$ are linearly dependent, contradicting the minimality of m. (As for $m \leq n$, any $n+1$ vectors in $\mathbb{C}^n$ are L.D.)

Thus $a_m \neq 0$

$$p(t) = t^m + \frac{a_{m-1}}{a_m} t^{m-1} + \cdots + \frac{a_0}{a_m}.$$

$$\implies p(A)\underline{y} = 0 \implies (0, \underline{y}) \text{ is an eigen pair of } p(A).$$
$$\implies \text{one of the zeros of } p(t) \text{ is an eigenvalue of } A.$$

Let that be λ.

$$p(t) = (t - \lambda)\, g(t) \qquad (g \text{ is an } m-1 \text{ degree polynomial}).$$

If $g(A)\underline{y} = 0$, minimality is a contradiction. So $g(A)\underline{y} \neq 0$. But

$$0 = p(A)\underline{y} = (A - \lambda I)\big(g(A)\underline{y}\big)$$

$\implies$ $g(A)\underline{y}$ is an eigenvector of A associated with the eigenvalue λ.

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• $\det(A - \lambda I)$ is a polynomial of degree n.

$\det(A - \lambda I) = 0$ is the *characteristic equation* where λ is the root.

Example:

$$A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) = 0 &\implies (7 - \lambda)(1 - \lambda) + 8 = 0 \\ &\implies \lambda^2 - 8\lambda + 15 = 0 \\ &\implies \lambda = 3,\ 5 \end{aligned}$$

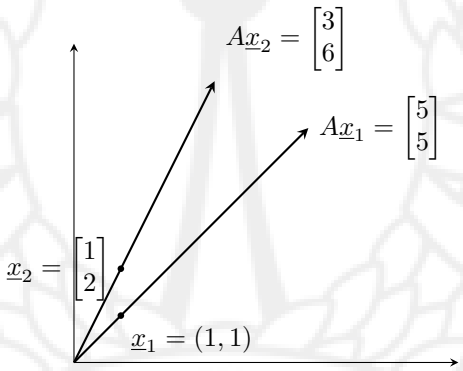
$$A - 3I = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix} \underline{x} = 0$$

$$\implies \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix} \underline{x} = 0$$

$$\implies \underline{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A - 5I = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \underline{x} = 0$$

$$\underline{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Spectrum of A:

Definition: The *spectrum* of $A \in M_n$ is the set of all $\lambda \in \mathbb{C}$ that are eigenvalues of A

$$\sigma(A) = \left\{\lambda \in \mathbb{C} \;\middle|\; A\underline{x} = \lambda\underline{x},\ \underline{x} \neq 0\right\}$$

- If $(\underline{x}, \lambda)$ is an eigenpair of A, then $(\alpha\underline{x}, \lambda)$ is also an eigenpair of A.
- How about $(S\underline{x}, \lambda)$ for a non-singular $S \in M_n$? This is not an eigen pair.

Polynomial of degree k:

$$p(t) = a_k t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0, \qquad a_k \neq 0$$

$p(t)$ at a matrix $A \in M_n$ can be expressed as

$$p(A) = a_k A^k + a_{k-1}A^{k-1} + \cdots + a_1 A + a_0 I.$$

If $a_k = 1$, we have a *monic* polynomial. $a_k^{-1}p(t)$ is always a monic polynomial.
Monic polynomial of degree $k \geq 1$ can be also represented as

$$p(t) = (t - \alpha_1)\cdots(t - \alpha_k)$$

If $p(\alpha_j) = 0$ for $j = 1, 2, \ldots, k$, then α_j is a root of $p(t) = 0$ or α_j is a zero of $p(t)$.

Also

$$p(A) = (A - \alpha_1 I) \cdots (A - \alpha_k I)$$

The eigenvalues of $p(A)$ are related to the eigenvalues of A.

Theorem: Let $p(t)$ be a given polynomial of degree k. If $\lambda, \underline{x}$ is an eigenvalue–eigenvector pair of $A \in M_n$, then $p(\lambda), \underline{x}$ is an eigenvalue–eigenvector pair of $p(A)$. Conversely, if $k \geq 1$ and μ is an eigenvalue of $p(A)$, then there is some eigenvalue λ of A such that $\mu = p(\lambda)$.

Proof: (a) We have

$$p(A)\underline{x} = a_k A^k \underline{x} + a_{k-1} A^{k-1} \underline{x} + \cdots + a_1 A\underline{x} + a_0 \underline{x}, \qquad a_k \neq 0$$

and

$$A^j \underline{x} = A^{j-1} A\underline{x} = A^{j-1} \lambda \underline{x} = \lambda A^{j-1} \underline{x} = \cdots = \lambda^j \underline{x}$$

$$\implies p(A)\underline{x} = a_k \lambda^k \underline{x} + \cdots + a_0 \underline{x} = (a_k \lambda^k + \cdots + a_0)\underline{x} = p(\lambda)\underline{x}$$

(b) Suppose μ is an eigenvalue of $p(A)$

$p(A) - \mu I$ is singular

$p(t)$ has degree $k \geq 1$.

$q(t) = p(t) - \mu$ has also degree $k \geq 1$

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$$q(t) = (t-\beta_1)(t-\beta_2)\cdots(t-\beta_k) \qquad \text{for } \beta_k \in \mathbb{C} \text{ or } \mathbb{R}$$

$$\begin{aligned} q(A) &= p(A) - \mu I \\ &= (A-\beta_1 I)\cdots(A-\beta_k I) \ \text{ is singular.} \end{aligned}$$

Some factor $(A-\beta_j I)$ is singular.

$$\begin{aligned} &\implies \beta_j \text{ is an eigenvalue of } A \\ &\quad 0 = q(\beta_j) = p(\beta_j) - \mu \\ &\implies \mu = p(\beta_j). \end{aligned}$$

$\lambda \in \sigma(A) \Rightarrow A - \lambda I$ singular. $\quad \lambda = 0$: $0 \in \sigma(A) \Rightarrow A - 0I$ singular $\Rightarrow A$ is singular. A is singular $\Rightarrow \det(A) = 0 \Rightarrow \det(A - 0I) = 0 \Rightarrow 0 \in \sigma(A)$.

- A matrix $A \in M_n$ is singular iff $0 \in \sigma(A)$. $\qquad (A\underline{x} = 0 \cdot \underline{x} = 0)$.

Theorem: $A \in M_n$ and $\lambda, \mu \in \mathbb{C}$. Then $\lambda \in \sigma(A)$ iff $\lambda + \mu \in \sigma(A + \mu I)$.

Proof: If $\lambda \in \sigma(A)$, $A\underline{x} = \lambda \underline{x}$ and hence

$$(A + \mu I)\underline{x} = A\underline{x} + \mu \underline{x} = (\lambda + \mu)\underline{x}.$$

Conversely, if $\lambda + \mu \in \sigma(A + \mu I)$

$$A\underline{y} + \mu \underline{y} = (A + \mu I)\underline{y} = (\lambda + \mu)\underline{y} \implies A\underline{y} = \lambda \underline{y}.$$

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Let $A \in M_n$. The multiplicity of an eigenvalue λ of A is its multiplicity as a zero of the characteristic polynomial $p_A(t)$. This is referred to as the *algebraic multiplicity*.

$$p_A(t) = \det(tI - A).$$

• The characteristic polynomial of each $A \in M_n$ has degree n and

$$p_A(t) = t^n - (\operatorname{tr} A)\, t^{n-1} + \cdots + (-1)^n \det(A)$$

If $\alpha_i \in \sigma(A)$

$$\begin{aligned} p_A(t) &= (t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n) \\ &= t^n - \sum_{i=1}^{n} \alpha_i\, t^{n-1} + \cdots + (-1)^n \prod_{i=1}^{n} \alpha_i \end{aligned}$$

$$\implies \operatorname{tr}(A) = \sum_{i=1}^{n} \alpha_i, \qquad \det(A) = \prod_{i=1}^{n} \alpha_i$$

Algebraic multiplicity: # of zeros in the characteristic equation

$$p_A(t) = (t-\alpha)^k q(t), \qquad q(\alpha) \neq 0$$

k is the multiplicity of zeros of $p_A(t)$, or the number of repeated roots of $p_A(t) = 0$. (*Thm. 1.2.13*)

Theorem: Let $A \in M_n$. There is some $\delta > 0$ such that $A + \epsilon I$ is nonsingular whenever $\epsilon \in \mathbb{C}$ and $0 < |\epsilon| < \delta$.

Proof:

$$\begin{aligned} \alpha \in \sigma(A) &\iff \lambda + \epsilon \in \sigma(A + \epsilon I) \\ &\implies 0 \in \sigma(A + \epsilon I) \iff \lambda + \epsilon = 0 \ \text{ for } \lambda \in \sigma(A) \\ &\implies \epsilon = -\lambda \ \text{ for some } \lambda \in \sigma(A). \end{aligned}$$

If all the eigenvalues of A are zero, take $\delta = 1$; else let $\delta = \min\{|\lambda| : \lambda \in \sigma(A) \text{ and } \lambda \neq 0\}$ $\implies -\epsilon \notin \sigma(A)$, $0 \notin \sigma(A + \epsilon I)$.

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Similarity

Def^n: Let $A, B \in M_n$ be given. We say that B is *similar* to A if there exists a nonsingular $S \in M_n$ such that

$$B = S^{-1}AS$$

Theorem: Let $A, B \in M_n$. If B is similar to A, then A and B have the same characteristic polynomial.

$$\begin{aligned} p_B(t) &= \det(tI - B) \\ &= \det(tS^{-1}S - S^{-1}AS) = \det\big(S^{-1}(tI - A)S\big) \\ &= \det S^{-1} \det(tI - A) \det S = (\det S)^{-1}(\det S) \det(tI - A) \\ &= \det(tI - A) = p_A(t). \end{aligned}$$

Corollary:

(a) A and B have the same eigenvalues.

(b) If B is a diagonal matrix, its main diagonal entries are the eigenvalues of A.

(c) $B = 0$ (a diagonal matrix) iff $A = 0$.

(d) $B = I$ (a diagonal matrix) iff $A = I$.

Def^n: If $A \in M_n$ is similar to a diagonal matrix, then A is said to be *diagonalizable*.

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Theorem: Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be $k \geq 2$ distinct eigenvalues of $A \in M_n$ (that is $\lambda_i \neq \lambda_j$ if $i \neq j$ and $1 \leq i, j \leq k$), and suppose that $\underline{x}^{(i)}$ is an eigenvector associated with λ_i for each $i = 1, \ldots, k$, then the vectors $\underline{x}^{(1)}, \ldots, \underline{x}^{(k)}$ are l.I.

Proof: For complex scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\alpha_1 \underline{x}^{(1)} + \alpha_2 \underline{x}^{(2)} + \cdots + \alpha_k \underline{x}^{(k)} = 0$$

Let $B_1 = (A - \lambda_2 I)(A - \lambda_3 I) \cdots (A - \lambda_k I)$

for eigenpair $(\lambda_i, \underline{x}^{(i)})$ for $i = 1, 2, \ldots, k$, we have

$$\begin{aligned} B_1 \underline{x}^{(i)} &= (\lambda_i - \lambda_2)(\lambda_i - \lambda_3) \cdots (\lambda_i - \lambda_k)\, \underline{x}^{(i)} \\ &= \begin{cases} 0 & 2 \leq i \leq k \\ \neq 0 & i = 1 \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} 0 = B_1\big(\alpha_1 \underline{x}^{(1)} + \alpha_2 \underline{x}^{(2)} + \cdots + \alpha_k \underline{x}^{(k)}\big) &= \alpha_1 B_1 \underline{x}^{(1)} + \alpha_2 B_1 \underline{x}^{(2)} + \cdots + \alpha_k B_1 \underline{x}^{(k)} \\ &= \alpha_1 B_1 \underline{x}^{(1)} + 0 + \cdots + 0 = \alpha_1 B_1 \underline{x}^{(1)} \end{aligned}$$

$$B_1 \underline{x}^{(1)} \neq 0 \implies \alpha_1 = 0$$

Similarly $\alpha_2 = \cdots = \alpha_k = 0$

$\implies \underline{x}^{(1)}, \ldots, \underline{x}^{(k)}$ are linearly independent.

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Diagonalizability: If $A \in M_n$ has n distinct eigenvalues, then A is diagonalizable. (Converse not true).

Two matrices are similar $\Rightarrow$ they share the same characteristic polynomial $\Rightarrow$ zeros of polynomial are same.

$$\implies \text{Eigenvalues are same.}$$

If the eigenvalues are not distinct, the characteristic poly^n:

$$p_A(t) = (t-\alpha)^k q(t).$$

k is algebraic multiplicity of eigenvalue of A.

Simultaneously Diagonalizable:

Def^n: $A, B \in M_n$ are *simultaneously diagonalizable* if $\exists$ a single non-singular $S \in M_n$ s.t. $S^{-1}AS$ and $S^{-1}BS$ are diagonal.

Claim: $A, B \in M_n$ be diagonalizable. Then A and B commute, i.e. $AB = BA$ iff they are simultaneously diagonalizable.

Proof: (a) Let say they are simultaneously diagonalizable

$$\begin{aligned}
&\implies S^{-1}AS = \Lambda_1 \quad \text{and} \quad S^{-1}BS = \Lambda_2 \\
&\implies \Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 \\
&\implies S^{-1}AS\,S^{-1}BS = S^{-1}BS\,S^{-1}AS \\
&\implies S^{-1}ABS = S^{-1}BAS \implies AB = BA \implies \text{commute}
\end{aligned}$$

(b) Suppose $AB = BA$ (after diagonalizing A, with equal eigenvalues grouped together)

$$A = \begin{bmatrix} \mu_1 I_{n_1} & & & \\ & \mu_2 I_{n_2} & & \\ & & \ddots & \\ & & & \mu_d I_{n_d} \end{bmatrix}$$ where $\mu_1, \ldots, \mu_d$ are the eigenvalues with algebraic multiplicity $n_1, n_2, \ldots, n$

Since $AB = BA$

$$B = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_d \end{bmatrix}, \qquad \text{each } B_i \in M_{n_i}$$

$\underline{x} \in E_{\lambda_i}(A) \Rightarrow A\underline{x} = \lambda_i \underline{x} \Rightarrow A(B\underline{x}) = AB\underline{x} = BA\underline{x} = B\lambda_i\underline{x} = \lambda_i(B\underline{x})$, so $B\underline{x} \in E_{\lambda_i}(A) \Rightarrow$ every eigenspace is invariant under $B \Rightarrow B$ cannot move a vector from one eigenspace to another.

each B_i is diagonalizable $\Rightarrow T_i^{-1} B_i T_i$ is diagonal. Then $T_i^{-1}\mu_i I_{n_i} T_i = \mu_i I_{n_i}$, so $T^{-1}AT = A$ and $T^{-1}BT$ are both diagonal.

Theorem: Suppose that $A \in M_{m,n}$ and $B \in M_{n,m}$ with $m \leq n$. Then the n eigenvalues of BA are the m eigenvalues of AB together with $n - m$ zeros; that is $p_{BA}(t) = t^{n-m} p_{AB}(t)$. If $m = n$ and at least one of A or B is nonsingular, then AB and BA are similar.

Proof:

$$\begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0_n \end{bmatrix} \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} 0_m & 0 \\ B & BA \end{bmatrix}$$

$C_1 = \begin{bmatrix} AB & 0 \\ B & 0_n \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0_m & 0 \\ B & BA \end{bmatrix}$ are similar.

The eigenvalues of C_1 are the eigenvalues of AB together with n zeros. The eigenvalues of C_2 are the eigenvalues of BA together with m zeros. $AB = A(BA)A^{-1}$ if A is non-singular and $m = n$.

Transpose & Hermitian: Let $A \in M_n$. Then the eigenvalues of A and A^T are the same. The eigenvalues of A and A^H are complex conjugate pairs.

$$\implies \det(tI - A) = \det(tI - A)^T = \det(tI - A^T).$$
$$\implies \det(\bar{t}I - A^H) = \det(tI - A)^H = \overline{\det(tI - A)}$$

$$p_{A^H}(\bar{t}) = \overline{p_A(t)}$$

and $p_{A^H}(\bar{\lambda}) = 0$ iff $p_A(\lambda) = 0$

Eigenspace of A: Let $A \in M_n$. For a given $\lambda \in \sigma(A)$, the set of all vectors $\underline{x} \in \mathbb{C}^n$ satisfying $A\underline{x} = \lambda\underline{x}$ is called the *eigenspace* of A associated with the eigenvalue λ. Every nonzero element of this eigenspace is an eigenvector of A associated with λ.

The dimension of the eigenspace of A associated with λ is the *geometric multiplicity* of λ.

λ is *simple* if its algebraic multiplicity is 1.

λ is *semisimple* if both are equal.

(a) $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ Geometric multiplicity = algebraic mult. = 1 (Simple)

$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Geometric multiplicity = algebraic multiplicity = 2 (Semisimple)

$A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Geometric multiplicity = 1, Algebraic multiplicity = 2.

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Algebraic multiplicity $(\lambda) \geq$ Geometric multiplicity (λ)

$$\implies \text{Alg. mult}(\lambda) > \text{Geo. mult.}(\lambda): \ A \text{ is defective.}$$
$$\implies \text{Alg. mult}(\lambda) = \text{Geo. mult.}(\lambda) \ \ \forall \lambda: \ A \text{ is non-defective}$$

Left-Eigenvector: A nonzero vector $\underline{y} \in \mathbb{C}^n$ is a *left eigenvector* of $A \in M_n$ associated with an eigenvalue λ of A if

$$\underline{y}^H A = \lambda \underline{y}^H.$$

Observation: Let $\underline{x} \in \mathbb{C}^n$ be nonzero, let $A \in M_n$ and suppose that $A\underline{x} = \lambda \underline{x}$. If $\underline{x}^H A = \mu \underline{x}^H$ then $\lambda = \mu$.

Proof: Let $\underline{x}$ is unit vector,

$$\mu = \mu \underline{x}^H \underline{x} = (\underline{x}^H A)\underline{x} = \underline{x}^H A \underline{x} = \underline{x}^H (A\underline{x}) = \underline{x}^H (\lambda \underline{x}) = \lambda \underline{x}^H \underline{x} = \lambda$$

Theorem: Let $A \in M_n$, nonzero vectors $\underline{x}, \underline{y} \in \mathbb{C}^n$, and scalars $\lambda, \mu \in \mathbb{C}$ be given. Suppose that $A\underline{x} = \lambda \underline{x}$ and $\underline{y}^H A = \mu \underline{y}^H$.

(a) If $\lambda \neq \mu$, then $\underline{y}^H \underline{x} = 0$

(b) If $\lambda = \mu$ and $\underline{y}^H \underline{x} \neq 0$, then there is a nonsingular $S \in M_n$ of the form $S = [\underline{x} \ \ S_1]$ such that

$$S^{-H} = \left[\ \underline{y}/(\underline{x}^H \underline{y}) \ \ Z_1 \ \right] \quad \text{and} \quad A = S \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} S^{-1}, \qquad B \in M_{n-1}$$

Proof: (a) Let $\underline{y}$ be a left eigenvector of A associated with μ and $\underline{x}$ be a right eigenvector of A associated with λ.

Now

$$\begin{aligned} \underline{y}^H A\underline{x} &= \underline{y}^H(\lambda\underline{x}) = \lambda(\underline{y}^H\underline{x}) \\ \implies (\mu\underline{y}^H)\underline{x} &= \lambda(\underline{y}^H\underline{x}) \end{aligned}$$

Since $\lambda \neq \mu$, $\lambda\underline{y}^H\underline{x} = \mu\underline{y}^H\underline{x}$ only if $\underline{y}^H\underline{x} = 0$.

Theorem: Let $A, B \in M_n$ and $B = S^{-1}AS$ for some non-singular S. If $\underline{x} \in \mathbb{C}^n$ is a right eigenvector of B associated with an eigenvalue λ, then $S\underline{x}$ is a right eigenvector of A associated with λ. If $\underline{y} \in \mathbb{C}^n$ is a left eigenvector of B associated with λ, then $S^{-H}\underline{y}$ is a left eigenvector of A associated with λ.

Proof: If $B\underline{x} = \lambda\underline{x}$, then $S^{-1}AS\underline{x} = \lambda\underline{x}$, or $A(S\underline{x}) = \lambda(S\underline{x})$. Since S is nonsingular and $\underline{x} \neq 0$, $S\underline{x} \neq 0$, and hence $S\underline{x}$ is an eigenvector of A. If $\underline{y}^H B = \lambda\underline{y}^H$, then

$$\underline{y}^H S^{-1}AS = \lambda\underline{y}^H \quad \text{or} \quad \left(S^{-H}\underline{y}\right)^H A = \lambda\left(S^{-H}\underline{y}\right)^H.$$

Hermitian matrices: If

$$A = A^H$$

Eigenvalues of real symmetric or complex Hermitian matrices are real:

$$\begin{aligned} A\underline{x} = \lambda\underline{x} &\implies \underline{x}^H A\underline{x} = \lambda\underline{x}^H\underline{x} \qquad ① \\ \implies \overline{\underline{x}^H A\underline{x}} = \bar{\lambda}\,\overline{\underline{x}^H\underline{x}} &\implies \underline{x}^H A^H\underline{x} = \bar{\lambda}\underline{x}^H\underline{x} \\ &\implies \underline{x}^H A\underline{x} = \bar{\lambda}\underline{x}^H\underline{x} \qquad ② \end{aligned}$$

① and ② $\implies \lambda = \bar{\lambda}$.