E2 212: Matrix Theory

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Problem Set 1

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1. Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$. Let $\mathcal{X}_i$, for $i = 1, 2, \ldots, K$, be a nonempty family of subspaces of $\mathcal{V}$. Show that
$$\mathcal{X} = \bigcap_{i=1}^{K} \mathcal{X}_i$$
is a subspace of $\mathcal{V}$.

2. For a vector space $\mathcal{V}$, and for $\mathcal{M}, \mathcal{N} \subseteq \mathcal{V}$, explain why
$$\text{span}(\mathcal{M} \cup \mathcal{N}) = \text{span}(\mathcal{M}) + \text{span}(\mathcal{N}).$$

3. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a vector space $\mathcal{V}$.
   (a) Prove that the intersection
   $$\mathcal{X} \cap \mathcal{Y}$$
   is also a subspace of $\mathcal{V}$.
   (b) Show that the union
   $$\mathcal{X} \cup \mathcal{Y}$$
   need not be a subspace of $\mathcal{V}$.

4. Let us denote the set of all $n \times n$ matrices over a field $\mathbb{F}$ by $\mathcal{M}_n(\mathbb{F})$.
   Let $\mathcal{S}_n$ denote the set of all $n \times n$ symmetric matrices, i.e.,
   $$A \in \mathcal{S}_n \quad \iff \quad A = A^T.$$
   Let $\mathcal{K}_n$ denote the set of all $n \times n$ skew-symmetric matrices, i.e.,
   $$B \in \mathcal{K}_n \quad \iff \quad B = -B^T.$$
   (a) Show that $\mathcal{S}_n$ is a subspace of $\mathcal{M}_n(\mathbb{F})$.
   (b) Show that $\mathcal{K}_n$ is a subspace of $\mathcal{M}_n(\mathbb{F})$.
   (c) Show that
   $$\mathcal{M}_n(\mathbb{F}) = \mathcal{S}_n \oplus \mathcal{K}_n.$$

5. If the product of two matrices is the zero matrix,
   $$AB = 0,$$
   show that the column space of B is contained in the null space of A; that is,
   $$\mathcal{R}(B) \subseteq \mathcal{N}(A).$$

6. If V and W are orthogonal subspaces, show that the only vector they have in common is the zero vector; that is,
$$V \cap W = \{0\}.$$

7. Prove that any set of nonzero mutually orthogonal vectors is linearly independent.

8. For matrices such that the product AB is defined, explain why each of the following statements is true:

   (a)
   $$\mathcal{R}(AB) \subseteq \mathcal{R}(A).$$

   (b)
   $$\mathcal{N}(AB) \supseteq \mathcal{N}(B).$$

9. Suppose that
$$\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$$
is a spanning set for $\mathcal{R}(B)$. Prove that
$$A(\mathcal{B}) = \{Ab_1, Ab_2, \ldots, Ab_n\}$$
is a spanning set for $\mathcal{R}(AB)$.

10. If $\mathcal{M}$ and $\mathcal{N}$ are subsets of a vector space $\mathcal{V}$, explain why
$$\begin{aligned}\dim(\text{span}(\mathcal{M} \cup \mathcal{N})) = &\dim(\text{span}(\mathcal{M})) + \dim(\text{span}(\mathcal{N})) \\ &- \dim(\text{span}(\mathcal{M}) \cap \text{span}(\mathcal{N}))\,.\end{aligned}$$

11. Let $\mathcal{X}$ be a plane passing through the origin in $\mathbb{R}^3$, and let $\mathcal{Y}$ be the line through the origin that is perpendicular to $\mathcal{X}$.

    Determine
    $$\mathcal{X} + \mathcal{Y}.$$

    Justify your answer.

12. For a general vector space, let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces with respective bases
$$\mathcal{B}_{\mathcal{X}} = \{x_1, x_2, \ldots, x_m\}$$
and
$$\mathcal{B}_{\mathcal{Y}} = \{y_1, y_2, \ldots, y_n\}.$$

    (a) Prove that
    $$\mathcal{X} \cap \mathcal{Y} = \{0\}$$
    if and only if
    $$\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$
    is a linearly independent set.

(b) Does

$$\mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}}$$

being linearly independent imply

$$\mathcal{X} \cap \mathcal{Y} = \{0\}?$$

Justify your answer.

(c) If

$$\mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}}$$

is a linearly independent set, does it follow that $\mathcal{X}$ and $\mathcal{Y}$ are complementary subspaces? Why?