E2 212: Matrix Theory

Aug–Dec 2026

Problem Set 2

(**Note**: $M_n(\mathbb{F})$ denotes the vector space of all $n \times n$ matrices whose elements come from a field $\mathbb{F}$)

1. Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m \times n}$. Let $\text{rank}(\mathbf{A}) = r$, and $\text{rank}(\mathbf{B}) = k$, prove that:

   (a) $r - k \leq \text{rank}(\mathbf{A} + \mathbf{B}) \leq r + k$, if $k \leq r$

   (b) $|r - k| \leq \text{rank}(\mathbf{A} - \mathbf{B})$

2. Let $\mathbf{S} \in M_n(\mathbb{F})$. For a vector space $\mathcal{V}$, define $\langle \cdot, \cdot \rangle_{\mathbf{S}} : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ by

   $$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{S}} = \langle \mathbf{S}\mathbf{x}, \mathbf{S}\mathbf{y} \rangle$$

   for $\mathbf{x}, \mathbf{y} \in \mathcal{V}$. Under what condition on $\mathbf{S}$ is $\langle \cdot, \cdot \rangle_{\mathbf{S}}$ an inner product on $\mathcal{V}$.

3. Cauchy–Schwarz inequality: Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space $\mathcal{V}$ over the field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Show that

   $$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathcal{V}$$

   with equality if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent, that is, if and only if $\mathbf{x} = \alpha \mathbf{y}$ or $\mathbf{y} = \alpha \mathbf{x}$ for some $\alpha \in \mathbb{F}$.

4. If $\langle \cdot, \cdot \rangle$ is an inner product on a real or complex vector space $\mathcal{V}$, then prove that the function $\| \cdot \| : \mathcal{V} \to [0, \infty)$ defined by $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ is a norm on $\mathcal{V}$.

5. Let $\langle \cdot, \cdot \rangle$ be a semi-inner product on a vector space $\mathcal{V}$ over the field $\mathbb{F}$. Show that

   $$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \text{ for all } \mathbf{x}, \mathbf{y} \in \mathcal{V}$$

   and the function $\| \cdot \| : \mathcal{V} \to [0, \infty)$ defined by $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ is a seminorm on $\mathcal{V}$.

6. Let $\| \cdot \|$ be a norm on $\mathcal{V}$ that is derived from an inner product. Show that it satisfies the *parallelogram identity*

   $$\frac{1}{2}(\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

   for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

7. Show that any seminorm on $\mathbb{C}^n$ is of the form $\| \cdot \|_S$ for some norm $\| \cdot \|$ and some $S \in M_n$.

8. Show that

   $$\big| \|\mathbf{x}\| - \|\mathbf{y}\| \big| \leq \|\mathbf{x} - \mathbf{y}\|$$

   for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

9. The triangle inequality $\|\mathbf{x}+\mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$ for a general ℓ_p-norm is the classical *Minkowski inequality*, which states that for $p \geq 1$,

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

Derive Minkowski's inequality. Hint: For $p > 1$, let q be the number such that $1/q = 1 - 1/p$. Verify that for scalars α and β,

$$|\alpha + \beta|^p = |\alpha + \beta||\alpha + \beta|^{p/q} \leq |\alpha||\alpha + \beta|^{p/q} + |\beta||\alpha + \beta|^{p/q},$$

and make use of Hölder's inequality.
The classical form of Hölder's inequality states that if $p > 1$ and $q > 1$ are real numbers such that $1/p + 1/q = 1$, then

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}.$$

10. Show that $\|\mathbf{x}\|_\infty = \lim_{p\to\infty} \|\mathbf{x}\|_p$.

11. Show that $\|\mathbf{x}\|_\infty$ is not derived from an inner product.

12. Let $\mathbf{S} \in M_{m,n}$ have full column rank, so $m \geq n$. Let $\|\cdot\|$ be a given norm on $\mathbb{C}^m$ and define

$$\|\mathbf{x}\|_S = \|\mathbf{S}\mathbf{x}\|$$

for $x \in \mathbb{C}^n$. Show that $\|\cdot\|_S$ is a norm on $\mathbb{C}^n$.