E2 212: Matrix Theory

Aug–Dec 2026

Problem Set 3

(**Note**: $M_n(\mathbb{F})$ denotes the vector space of all $n \times n$ matrices whose elements come from a field $\mathbb{F}$)

1. Let $f(\mathbf{x}) = \|\mathbf{x}\|_1$ be the ℓ_1-norm on $\mathbb{C}^3$, and let $f^D(\cdot)$ denote its dual norm.
   Given $\mathbf{y} = [3, 1, 2]^T$:

   (a) Find a vector $\mathbf{x} \in \mathbb{C}^3$ that solves the optimization problem:

   $$\max_{\|\mathbf{x}\|_1 = 1} \quad |\mathbf{y}^H \mathbf{x}|$$

   (b) Compute $f^D(\mathbf{y})$ by substituting x.

2. If $\mathbf{A}, \mathbf{B} \in M_n$, $\mathbf{A}$ is invertible, and if $\mathbf{A} + \mathbf{B}$ is singular, show that

   $$|||\mathbf{B}||| \geq \frac{1}{|||\mathbf{A}^{-1}|||}$$

   for any matrix norm $|||\cdot|||$.

3. Derive the matrix norm induced by the ℓ_∞-norm on $\mathbb{C}^n$.

4. Let $|||\cdot|||_p$ denote the matrix norm induced by the ℓ_p-vector norm, and $\|\cdot\|_p$ denote the ℓ_p vector norm on matrices, then prove the following for $\mathbf{A} \in M_n$:

   (i) $|||\mathbf{A}|||_1 \leq \|\mathbf{A}\|_1 \leq n|||\mathbf{A}|||_\infty$.
   (ii) $|||\mathbf{A}|||_1 \leq \sqrt{n}\|\mathbf{A}\|_2$.

5. Let

   $$\rho(\mathbf{A}) = \max\{|\lambda| : \mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \forall \mathbf{x} \neq 0\}$$

   be the spectral radius of $\mathbf{A} \in M_n$. Suppose $|||\cdot|||$ be a matrix norm on M_n. Then show that

   $$|\lambda| \leq \rho(\mathbf{A}) \leq |||\mathbf{A}|||$$

   where λ is an eigenvalue of $\mathbf{A}$.

6. Let $|||\cdot|||$ be any matrix norm on M_n and let $\mathbf{A} \in M_n$. Then show that the spectral radius of $\mathbf{A}$ defined as $\rho(\mathbf{A}) = \max\{|\lambda| : \mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \forall \mathbf{x} \in \mathbb{C}^n\}$ can be obtained as

   $$\rho(\mathbf{A}) = \lim_{k\to\infty} \left|\left|\left|\mathbf{A}^k\right|\right|\right|^{1/k}$$

7. A matrix $\mathbf{A} \in M_n$ is non-singular if there is a matrix norm $|||\cdot|||$ such that $|||\mathbf{I} - \mathbf{A}||| < 1$. Determine $\mathbf{A}^{-1}$.

8. Show that for any $\mathbf{A} \in \mathbb{R}^{n\times n}$, the series $\sum_{k=1}^{\infty} a_k \mathbf{A}^k$ converges if there is a matrix norm $|||\cdot|||$ on M_n such that the numerical series $\sum_{k=0}^{\infty} |a_k| \cdot |||\mathbf{A}|||^k$ converges. (Hint: what does convergence of a series mean?)

9. Show that the dual norm $\|\cdot\|^D$ of the vector norm $\|\cdot\|$ on $\mathbb{C}^n$ induces the matrix norm $|||(\cdot)^H|||$. Here, $\mathbf{A}^H$ denotes complex conjugate transpose of $\mathbf{A}$.

10. Let $|||\mathbf{A}|||_2$ be the spectral norm, which is the matrix norm induced by the ℓ_2-norm on $\mathbb{C}^n$. Show that
$$\min_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \frac{1}{|||\mathbf{A}^{-1}|||_2}.$$

11. Let $\mathbf{A} \in M_n(\mathbb{R})$ be an invertible matrix and $\mathbf{B} \in M_n(\mathbb{R})$ be a non-invertible matrix. Show that (use the result in question 10)
$$|||\mathbf{A}-\mathbf{B}|||_2 \geq \frac{1}{|||\mathbf{A}^{-1}|||_2}.$$
(*Hint:* Since $\mathbf{B}$ is non-invertible, $\exists$ a non-zero $\mathbf{x}$ so that $\mathbf{B}\mathbf{x} = \mathbf{0}$.)