E2 212: Matrix Theory

PROBLEM SET 1 SOLUTIONS

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Problem Set 1 – Solutions

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1. Let V be a vector space over a field F. Let X_i, for $i = 1, 2, \ldots, K$, be a nonempty family of subspaces of V. Show that

$$X = \bigcap_{i=1}^{K} X_i$$

is a subspace of V.

Solution. Since each X_i is a subspace, $0 \in X_i$ for every i. Hence

$$0 \in \bigcap_{i=1}^{K} X_i = X,$$

so X is nonempty.

Let $x, y \in X$. Then $x, y \in X_i$ for every i. Since each X_i is closed under addition,

$$x + y \in X_i \qquad \text{for every } i,$$

and therefore $x + y \in X$.

Now let $\alpha \in F$ and $x \in X$. Then $x \in X_i$ for every i. Since each X_i is closed under scalar multiplication,

$$\alpha x \in X_i \qquad \text{for every } i,$$

and hence $\alpha x \in X$.

Thus X contains 0 and is closed under addition and scalar multiplication. Therefore X is a subspace of V.

2. For a vector space V, and for $M, N \subseteq V$, explain why

$$\operatorname{span}(M \cup N) = \operatorname{span}(M) + \operatorname{span}(N).$$

Solution. We prove both inclusions.

First, let $x \in \operatorname{span}(M \cup N)$. Then x is a finite linear combination of vectors from $M \cup N$. Grouping together the terms coming from M and those coming from N, we may write

$$x = u + v,$$

where $u \in \operatorname{span}(M)$ and $v \in \operatorname{span}(N)$. Hence

$$x \in \operatorname{span}(M) + \operatorname{span}(N),$$

so

$$\operatorname{span}(M \cup N) \subseteq \operatorname{span}(M) + \operatorname{span}(N).$$

Conversely, let $x \in \operatorname{span}(M) + \operatorname{span}(N)$. Then $x = u + v$ for some $u \in \operatorname{span}(M)$ and $v \in \operatorname{span}(N)$. The vector u is a finite linear combination of vectors in M, and v is a finite linear combination of vectors in N. Therefore $u + v$ is a finite linear combination of vectors in $M \cup N$. Thus

$$x \in \operatorname{span}(M \cup N).$$

Hence

$$\operatorname{span}(M) + \operatorname{span}(N) \subseteq \operatorname{span}(M \cup N).$$

Therefore,

$$\boxed{\operatorname{span}(M \cup N) = \operatorname{span}(M) + \operatorname{span}(N)}.$$

3. Let X and Y be two subspaces of a vector space V.

(a) Prove that the intersection

$$X \cap Y$$

is also a subspace of V.

Solution. Since X and Y are subspaces, $0 \in X$ and $0 \in Y$, so $0 \in X \cap Y$.

Let $u, v \in X \cap Y$. Then $u, v \in X$ and $u, v \in Y$. Since both X and Y are closed under addition,

$$u + v \in X \quad \text{and} \quad u + v \in Y.$$

Thus $u + v \in X \cap Y$.

Similarly, if $\alpha \in F$ and $u \in X \cap Y$, then $u \in X$ and $u \in Y$. Hence

$$\alpha u \in X \quad \text{and} \quad \alpha u \in Y,$$

so $\alpha u \in X \cap Y$.

Therefore $X \cap Y$ is a subspace of V.

(b) Show that the union

$$X \cup Y$$

need not be a subspace of V.

Solution. Consider $V = \mathbb{R}^2$ with

$$X = \text{span}\{(1, 0)\}, \qquad Y = \text{span}\{(0, 1)\}.$$

Then $(1, 0) \in X \subseteq X \cup Y$ and $(0, 1) \in Y \subseteq X \cup Y$. However,

$$(1, 0) + (0, 1) = (1, 1),$$

and $(1, 1)$ belongs to neither X nor Y. Hence

$$(1, 1) \notin X \cup Y.$$

Thus $X \cup Y$ is not closed under addition, so it need not be a subspace of V.

4. Let us denote the set of all $n \times n$ matrices over a field F by $M_n(F)$. Let S_n denote the set of all $n \times n$ symmetric matrices, i.e.,

$$A \in S_n \quad \Longleftrightarrow \quad A = A^T.$$

Let K_n denote the set of all $n \times n$ skew-symmetric matrices, i.e.,

$$B \in K_n \quad \Longleftrightarrow \quad B = -B^T.$$

(a) Show that S_n is a subspace of $M_n(F)$.

Solution. The zero matrix is symmetric, since $0^T = 0$. Thus $0 \in S_n$.

If $A, B \in S_n$, then $A^T = A$ and $B^T = B$. Hence

$$(A + B)^T = A^T + B^T = A + B,$$

so $A + B \in S_n$.

If $c \in F$ and $A \in S_n$, then

$$(cA)^T = cA^T = cA,$$

so $cA \in S_n$. Therefore S_n is a subspace of $M_n(F)$.

(b) Show that K_n is a subspace of $M_n(F)$.

Solution. The zero matrix is skew-symmetric, since $0^T = 0 = -0$. Thus $0 \in K_n$.

If $A, B \in K_n$, then $A^T = -A$ and $B^T = -B$. Therefore

$$(A+B)^T = A^T + B^T = -A - B = -(A+B),$$

so $A + B \in K_n$.

If $c \in F$ and $A \in K_n$, then

$$(cA)^T = cA^T = -cA,$$

so $cA \in K_n$. Therefore K_n is a subspace of $M_n(F)$.

(c) Show that

$$M_n(F) = S_n \oplus K_n.$$

Solution. For this part, we assume $\mathrm{char}(F) \neq 2$, so that $\frac{1}{2}$ is defined in F. This assumption is necessary for the statement as written.

Let $A \in M_n(F)$. Write

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

Set

$$S = \frac{1}{2}(A + A^T), \qquad K = \frac{1}{2}(A - A^T).$$

Then

$$S^T = \frac{1}{2}(A^T + A) = S,$$

so $S \in S_n$, and

$$K^T = \frac{1}{2}(A^T - A) = -K,$$

so $K \in K_n$. Hence every $A \in M_n(F)$ belongs to $S_n + K_n$, and therefore

$$M_n(F) = S_n + K_n.$$

It remains to show that the sum is direct. Let $A \in S_n \cap K_n$. Then

$$A^T = A \qquad \text{and} \qquad A^T = -A.$$

Thus $A = -A$, so $2A = 0$. Since $\mathrm{char}(F) \neq 2$, it follows that $A = 0$. Hence

$$S_n \cap K_n = \{0\}.$$

Therefore

$$\boxed{M_n(F) = S_n \oplus K_n}.$$

If $\mathrm{char}(F) = 2$, then $-A = A$ for every matrix A, so $K_n = S_n$ and the stated direct-sum conclusion is generally false.

5. If the product of two matrices is the zero matrix,

$$AB = 0,$$

show that the column space of B is contained in the null space of A; that is,

$$\mathcal{R}(B) \subseteq \mathcal{N}(A).$$

Solution. Let $y \in \mathcal{R}(B)$. By definition of the column space, there exists a vector x such that

$$y = Bx.$$

Then

$$Ay = A(Bx) = (AB)x = 0x = 0.$$

Thus $y \in \mathcal{N}(A)$. Since every $y \in \mathcal{R}(B)$ belongs to $\mathcal{N}(A)$,

$$\boxed{\mathcal{R}(B) \subseteq \mathcal{N}(A)}.$$

6. If V and W are orthogonal subspaces, show that the only vector they have in common is the zero vector; that is,

$$V \cap W = \{0\}.$$

Solution. Let $x \in V \cap W$. Then $x \in V$ and $x \in W$. Since V and W are orthogonal subspaces, every vector in V is orthogonal to every vector in W. In particular, because x belongs to both subspaces,

$$\langle x, x \rangle = 0.$$

By positive definiteness of the inner product, this implies $x = 0$. Hence

$$V \cap W \subseteq \{0\}.$$

Since both V and W are subspaces, $0 \in V \cap W$. Therefore

$$\boxed{V \cap W = \{0\}}.$$

7. Prove that any set of nonzero mutually orthogonal vectors is linearly independent.

Solution. Let $v_1, \ldots, v_k$ be nonzero mutually orthogonal vectors, and suppose

$$c_1 v_1 + \cdots + c_k v_k = 0.$$

Fix $j \in \{1, \ldots, k\}$ and take the inner product of both sides with v_j. Then

$$0 = \left\langle \sum_{i=1}^{k} c_i v_i, v_j \right\rangle = \sum_{i=1}^{k} c_i \langle v_i, v_j \rangle.$$

Because the vectors are mutually orthogonal, every term with $i \neq j$ is zero. Hence

$$0 = c_j \langle v_j, v_j \rangle = c_j \|v_j\|^2.$$

Since $v_j \neq 0$, we have $\|v_j\|^2 > 0$, and therefore $c_j = 0$. Since j was arbitrary,

$$c_1 = \cdots = c_k = 0.$$

Thus the vectors are linearly independent.

8. For matrices such that the product AB is defined, explain why each of the following statements is true:

(a)

$$\mathcal{R}(AB) \subseteq \mathcal{R}(A).$$

Solution. Let $y \in \mathcal{R}(AB)$. Then there exists a vector x such that

$$y = ABx = A(Bx).$$

Thus y is of the form Az with $z = Bx$, so $y \in \mathcal{R}(A)$. Hence

$$\boxed{\mathcal{R}(AB) \subseteq \mathcal{R}(A)}.$$

(b)

$$\mathcal{N}(AB) \supseteq \mathcal{N}(B).$$

Solution. Let $x \in \mathcal{N}(B)$. Then $Bx = 0$, and therefore

$$ABx = A(Bx) = A0 = 0.$$

Hence $x \in \mathcal{N}(AB)$. Thus

$$\boxed{\mathcal{N}(AB) \supseteq \mathcal{N}(B)}.$$

9. Suppose that

$$B = \{b_1, b_2, \ldots, b_n\}$$

is a spanning set for $\mathcal{R}(B)$. Prove that

$$A(B) = \{Ab_1, Ab_2, \ldots, Ab_n\}$$

is a spanning set for $\mathcal{R}(AB)$.

Solution. Let $y \in \mathcal{R}(AB)$. Then for some vector x,

$$y = ABx = A(Bx).$$

Since $Bx \in \mathcal{R}(B)$ and $\{b_1, \ldots, b_n\}$ spans $\mathcal{R}(B)$, there exist scalars $c_1, \ldots, c_n$ such that

$$Bx = c_1 b_1 + \cdots + c_n b_n.$$

Therefore

$$y = A(Bx) = c_1 Ab_1 + \cdots + c_n Ab_n.$$

Hence every vector in $\mathcal{R}(AB)$ lies in

$$\text{span}\{Ab_1, \ldots, Ab_n\}.$$

Also, because each $b_i \in \mathcal{R}(B)$, there exists x_i such that $b_i = Bx_i$. Therefore

$$Ab_i = ABx_i \in \mathcal{R}(AB).$$

Thus all the vectors Ab_i belong to $\mathcal{R}(AB)$ and together they span it. Hence

$$\boxed{A(B) = \{Ab_1, \ldots, Ab_n\} \text{ is a spanning set for } \mathcal{R}(AB)}.$$

10. If M and N are subsets of a vector space V, explain why

$$\dim(\text{span}(M \cup N)) = \dim(\text{span}(M)) + \dim(\text{span}(N)) - \dim(\text{span}(M) \cap \text{span}(N)).$$

Solution. Assume the relevant spans are finite-dimensional, as required for the subtraction formula above. Let

$$U = \text{span}(M), \qquad W = \text{span}(N).$$

From Question 2,

$$\text{span}(M \cup N) = U + W.$$

Let $\{z_1, \ldots, z_r\}$ be a basis of $U \cap W$. Extend it to a basis of U:

$$\{z_1, \ldots, z_r, u_1, \ldots, u_p\},$$

and to a basis of W:

$$\{z_1, \ldots, z_r, w_1, \ldots, w_q\}.$$

Then

$$\{z_1, \ldots, z_r, u_1, \ldots, u_p, w_1, \ldots, w_q\}$$

is a basis of $U + W$. Hence

$$\dim(U + W) = r + p + q.$$

But

$$\dim U = r + p, \qquad \dim W = r + q, \qquad \dim(U \cap W) = r.$$

Therefore

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Substituting $U = \operatorname{span}(M)$ and $W = \operatorname{span}(N)$ gives

$$\boxed{\dim(\operatorname{span}(M \cup N)) = \dim(\operatorname{span}(M)) + \dim(\operatorname{span}(N)) - \dim(\operatorname{span}(M) \cap \operatorname{span}(N))}.$$

11. Let X be a plane passing through the origin in $\mathbb{R}^3$, and let Y be the line through the origin that is perpendicular to X. Determine

$$X + Y.$$

Justify your answer.

Solution. Since X is a plane through the origin,

$$\dim X = 2.$$

Since Y is the line through the origin perpendicular to X,

$$\dim Y = 1.$$

Also $X \perp Y$, so by Question 6,

$$X \cap Y = \{0\}.$$

Therefore

$$\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y) = 2 + 1 - 0 = 3.$$

Since $X + Y$ is a 3-dimensional subspace of $\mathbb{R}^3$, it must be all of $\mathbb{R}^3$. Thus

$$\boxed{X + Y = \mathbb{R}^3}.$$

12. For a general vector space, let X and Y be two subspaces with respective bases

$$B_X = \{x_1, x_2, \ldots, x_m\}$$

and

$$B_Y = \{y_1, y_2, \ldots, y_n\}.$$

(a) Prove that

$$X \cap Y = \{0\}$$

if and only if

$$\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$

is a linearly independent set.

Solution. Here the displayed collection is understood as the concatenated family $x_1, \ldots, x_m, y_1, \ldots, y_n$.

Suppose first that $X \cap Y = \{0\}$. Assume

$$a_1x_1 + \cdots + a_mx_m + b_1y_1 + \cdots + b_ny_n = 0.$$

Then

$$a_1x_1 + \cdots + a_mx_m = -(b_1y_1 + \cdots + b_ny_n).$$

The left side belongs to X and the right side belongs to Y, so their common value belongs to $X \cap Y = \{0\}$. Hence

$$a_1x_1 + \cdots + a_mx_m = 0, \qquad b_1y_1 + \cdots + b_ny_n = 0.$$

Because B_X and B_Y are bases, each is linearly independent. Therefore

$$a_1 = \cdots = a_m = 0, \qquad b_1 = \cdots = b_n = 0.$$

Thus the concatenated family is linearly independent.

Conversely, suppose the concatenated family

$$x_1, \ldots, x_m, y_1, \ldots, y_n$$

is linearly independent. Let $z \in X \cap Y$. Since $z \in X$ and $z \in Y$, there exist scalars a_i and b_j such that

$$z = \sum_{i=1}^{m} a_ix_i = \sum_{j=1}^{n} b_jy_j.$$

Therefore

$$\sum_{i=1}^{m} a_ix_i - \sum_{j=1}^{n} b_jy_j = 0.$$

By linear independence of the concatenated family, all a_i and b_j are zero. Hence $z = 0$. Therefore

$$\boxed{X \cap Y = \{0\}}.$$

(b) Does

$$B_X \cup B_Y$$

being linearly independent imply

$$X \cap Y = \{0\}?$$

Justify your answer.

Solution. No. Literal set union removes repeated elements. For example, let

$$X = Y = \operatorname{span}\{e_1\}$$

with

$$B_X = B_Y = \{e_1\}.$$

Then

$$B_X \cup B_Y = \{e_1\}$$

is linearly independent, but

$$X \cap Y = X = \operatorname{span}\{e_1\} \neq \{0\}.$$

Thus the implication is false.

(c) If

$$B_X \cup B_Y$$

is a linearly independent set, does it follow that X and Y are complementary subspaces? Why?

Solution. No. Even if the union is linearly independent, it need not span the whole ambient space.

For example, in $V = \mathbb{R}^3$, let

$$X = \text{span}\{e_1\}, \qquad Y = \text{span}\{e_2\}.$$

Then

$$B_X \cup B_Y = \{e_1, e_2\}$$

is linearly independent, but

$$X + Y = \text{span}\{e_1, e_2\} \neq \mathbb{R}^3.$$

Hence X and Y are not complementary subspaces of V.