E2 212: Matrix Theory

PROBLEM SET 3 SOLUTIONS

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Note: $|||\cdot|||$ denotes a matrix norm (submultiplicative); $|||\cdot|||_p$ is the matrix norm induced by the ℓ_p vector norm; $\|A\|_p$ is the ℓ_p norm of A viewed as a vector of n^2 entries. $\rho(A) = \max\{|\lambda| : Ax = \lambda x \text{ for some } x \neq 0\}$. The dual norm of f is $f^D(y) = \max_{f(x)=1} |y^H x|$.

Problem 1

Here $f(x) = \|x\|_1$ and $y = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ is real, so for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3$,

$$y^H x = 3x_1 + x_2 + 2x_3.$$

(a) We need x such that

$$f^D(y) = \max_{\|x\|_1=1} |y^H x| = \max_{|x_1|+|x_2|+|x_3|=1} |3x_1 + x_2 + 2x_3|.$$

By the triangle inequality,

$$|3x_1 + x_2 + 2x_3| \leq 3|x_1| + |x_2| + 2|x_3|.$$

Since the quantity on the right is an upper bound that we can actually attain (shown below), we maximize it instead. Using the constraint $|x_1| + |x_2| + |x_3| = 1$,

$$3|x_1| + |x_2| + 2|x_3| = \underbrace{(|x_1| + |x_2| + |x_3|)}_{=1} + 2|x_1| + |x_3| = 1 + 2|x_1| + |x_3|.$$

Equivalently, since $3(|x_1| + |x_2| + |x_3|) = 3$,

$$3|x_1| + |x_2| + 2|x_3| = 3 - 2|x_2| - |x_3| \leq 3.$$

The maximum value 3 is attained if and only if $2|x_2| + |x_3| = 0$, i.e.

$$|x_2| = 0, \qquad |x_3| = 0 \implies |x_1| = 1.$$

(Any mass placed on x_2 or x_3 strictly reduces the value, because their weights $1, 2$ are smaller than the weight 3 of x_1.)

For $x = \begin{bmatrix} e^{j\theta} \\ 0 \\ 0 \end{bmatrix}$ we have $\|x\|_1 = 1$ and $|y^H x| = |3e^{j\theta}| = 3$, so the upper bound is attained (the triangle inequality is tight). Hence

$$\boxed{x = \begin{bmatrix} e^{j\theta} \\ 0 \\ 0 \end{bmatrix}, \quad 0 \leq \theta \leq 2\pi}$$

solves the optimization problem (and these are the only maximizers).

(b) Substituting x,

$$f^D(y) = |y^H x| = |3e^{j\theta}| = 3 \quad (= \|y\|_\infty).$$

■

Problem 2

Lemma (used here and in Problem 7)**.** *If $A \in M_n$ is singular, then $\|I - A\| \geq 1$ for every matrix norm.*

Proof. There exists $x \neq 0$ with $Ax = 0$. Then $(I - A)x = x$, so 1 is an eigenvalue of $I - A$ and $\rho(I - A) \geq 1$. By Problem 5, $\rho(I - A) \leq \|I - A\|$, hence $\|I - A\| \geq 1$. □

Proof of the claim: Write

$$A + B = A(I + A^{-1}B).$$

Since A is invertible and $A + B$ is singular, $I + A^{-1}B$ must be singular (a product of an invertible matrix and a singular one is singular; equivalently, $\det(A + B) = \det A \det(I + A^{-1}B) = 0$ with $\det A \neq 0$). Applying the Lemma to $M = I + A^{-1}B$,

$$\left\|I - (I + A^{-1}B)\right\| \geq 1 \implies \left\|-A^{-1}B\right\| = \left\|A^{-1}B\right\| \geq 1.$$

By submultiplicativity,

$$\left\|A^{-1}\right\| \|B\| \geq \left\|A^{-1}B\right\| \geq 1 \implies \|B\| \geq \frac{1}{\|A^{-1}\|}. \qquad \blacksquare$$

Problem 3

Matrix norm induced by ℓ_∞:

$$\|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{\|x\|_\infty = 1} \|Ax\|_\infty, \qquad \|Ax\|_\infty = \max_{1 \leq i \leq n} |(Ax)_i|.$$

Claim: $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$ (maximum absolute row sum).

Upper bound: For each i,

$$|(Ax)_i| = \Big| \sum_{j=1}^{n} a_{ij} x_j \Big| \leq \sum_{j=1}^{n} |a_{ij}||x_j| \leq \Big(\sum_{j=1}^{n} |a_{ij}| \Big) \|x\|_\infty.$$

Taking the maximum over i,

$$\|Ax\|_\infty \leq \Big(\max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \Big) \|x\|_\infty \implies \|A\|_\infty \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|. \tag{1}$$

Lower bound: Let k be a row at which the maximum row sum is attained (k is *not* the index of $\|\cdot\|_\infty$). Choose

$$x_j = \begin{cases} \dfrac{\overline{a_{kj}}}{|a_{kj}|}, & a_{kj} \neq 0, \\ 1, & a_{kj} = 0. \end{cases}$$

Then $|x_j| = 1$ for all j, so $\|x\|_\infty = 1$. Moreover $a_{kj}x_j = |a_{kj}|$ for every j (also when $a_{kj} = 0$), so

$$(Ax)_k = \sum_{j=1}^{n} a_{kj}x_j = \sum_{j=1}^{n} |a_{kj}|.$$

Therefore

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \;\geq\; |(Ax)_k| = \sum_{j=1}^{n} |a_{kj}| = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| \implies |||A|||_\infty \geq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|. \tag{2}$$

From (1) and (2),

$$\boxed{|||A|||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|} \qquad \blacksquare$$

Problem 4

Recall

$$|||A|||_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| \text{ (max column sum)}, \qquad |||A|||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| \text{ (max row sum)},$$

$$\|A\|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|, \qquad \|A\|_2 = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{1/2}.$$

(i)(a) $|||A|||_1 \leq \|A\|_1$: the largest column sum is at most the sum of all column sums,

$$|||A|||_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| \;\leq\; \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}| = \|A\|_1 .$$

(i)(b) $\|A\|_1 \leq n\,|||A|||_\infty$: every row sum is at most the maximum row sum. Let k be a row with the maximum row sum. Then

$$\|A\|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}| \;\leq\; \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{kj}| = n\sum_{j=1}^{n} |a_{kj}| = n\,|||A|||_\infty .$$

Hence $|||A|||_1 \leq \|A\|_1 \leq n\,|||A|||_\infty$.

(ii) $|||A|||_1 \leq \sqrt{n}\,\|A\|_2$. By the Cauchy–Schwarz inequality, for any $a \in \mathbb{C}^n$ (take the vector $(1,\ldots,1)^T$ against $(|a_1|,\ldots,|a_n|)^T$),

$$\sum_{i=1}^{n} |a_i| \;\leq\; \sqrt{n}\Big(\sum_{i=1}^{n} |a_i|^2\Big)^{1/2}.$$

Apply this to each column j of A:

$$\sum_{i=1}^{n} |a_{ij}| \;\leq\; \sqrt{n}\Big(\sum_{i=1}^{n} |a_{ij}|^2\Big)^{1/2} \;\leq\; \sqrt{n}\Big(\sum_{k=1}^{n}\sum_{i=1}^{n} |a_{ik}|^2\Big)^{1/2} = \sqrt{n}\,\|A\|_2 .$$

Taking the maximum over j gives $|||A|||_1 \leq \sqrt{n}\,\|A\|_2$. $\blacksquare$

Problem 5

(i) $|\lambda| \le \rho(A)$ for every eigenvalue λ of A — immediate from the definition of $\rho(A)$ as a maximum over the spectrum $\sigma(A)$.

(ii) $\rho(A) \le |||A|||$. Let $\lambda \in \sigma(A)$ with eigenvector $x \neq 0$. A general matrix norm need not come with a vector norm, so we avoid using $\|Ax\|$ and instead form the $n \times n$ matrix

$$X = \begin{bmatrix} x & x & \cdots & x \end{bmatrix} \neq 0.$$

Then $AX = \lambda X$, so

$$|\lambda| \, |||X||| = |||\lambda X||| = |||AX||| \le |||A||| \, |||X|||.$$

Since $|||X||| > 0$, we get $|\lambda| \le |||A|||$. As λ was an arbitrary eigenvalue,

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda| \le |||A|||.$$

(For an *induced* norm one may simply use $\|Ax\| = |\lambda| \, \|x\| \le |||A||| \, \|x\|$.)

Combining (i) and (ii): $|\lambda| \le \rho(A) \le |||A|||$. ■

Problem 6

We will prove two lemmas.

Lemma (1)**.** *If $A \in M_n$ and $|||A||| < 1$ for some matrix norm, then $\lim_{k\to\infty} A^k = 0$.*

Proof. By submultiplicativity $\left|\left|\left|A^k\right|\right|\right| \le |||A|||^k \to 0$ since $|||A||| < 1$. So $A^k \to 0$ with respect to $|||\cdot|||$. All norms on $\mathbb{C}^{n^2}$ are equivalent, hence $A^k \to 0$ with respect to $\|\cdot\|_\infty$, i.e. every entry of A^k tends to 0. □

Lemma (2)**.** *For $A \in M_n$: $\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1$.*

Proof. For ($\Rightarrow$), let $Ax = \lambda x$, $x \neq 0$. Then $A^k x = \lambda^k x \to 0$, which forces $|\lambda| < 1$. This holds for every $\lambda \in \sigma(A)$, so $\rho(A) < 1$.

For ($\Leftarrow$), let $\rho(A) < 1$. By Schur triangularization, $A = UTU^H$ with U unitary and T upper triangular with diagonal entries $\lambda_1, \ldots, \lambda_n$. For $t \in (0, 1]$ let $D_t = \mathrm{diag}(1, t, \ldots, t^{n-1})$. Then $(D_t^{-1} T D_t)_{ij} = t^{j-i} T_{ij}$ for $j \ge i$, so the maximum row sum satisfies

$$\left|\left|\left| D_t^{-1} T D_t \right|\right|\right|_\infty \le \rho(A) + t\, C, \qquad C = \max_i \sum_{j>i} |T_{ij}|.$$

Choose t small enough that this is < 1. With $S = UD_t$, the function $|||B|||' := \left|\left|\left| S^{-1} B S \right|\right|\right|_\infty$ is a matrix norm (it is submultiplicative), and $|||A|||' < 1$. By Lemma 1, $A^k \to 0$. □

Proof of $\rho(A) = \lim_{k\to\infty} \left|\left|\left| A^k \right|\right|\right|^{1/k}$:

Lower bound: If $Ax = \lambda x$, then $A^k x = \lambda^k x$, so λ^k is an eigenvalue of A^k. Hence $|\lambda|^k \le \rho(A^k)$ for all $\lambda \in \sigma(A)$, i.e. $(\rho(A))^k \le \rho(A^k)$. By Problem 5, $\rho(A^k) \le \|\!|A^k|\!\|$. Therefore

$$\rho(A) \le \|\!|A^k|\!\|^{1/k} \quad \text{for all } k \ge 1. \tag{1}$$

Upper bound: Fix $\varepsilon > 0$ and define

$$\widehat{A} = \frac{1}{\rho(A)+\varepsilon} A.$$

Then $\rho(\widehat{A}) = \dfrac{\rho(A)}{\rho(A)+\varepsilon} < 1$, so by Lemma 2, $\widehat{A}^k \to 0$, i.e. $\|\!|\widehat{A}^k|\!\| \to 0$. Hence there is $N \in \mathbb{N}$ such that $\|\!|\widehat{A}^k|\!\| < 1$ for all $k \ge N$:

$$\left\|\!\left| \frac{1}{(\rho(A)+\varepsilon)^k} A^k \right|\!\right\| < 1 \implies \|\!|A^k|\!\| < (\rho(A)+\varepsilon)^k \implies \|\!|A^k|\!\|^{1/k} < \rho(A)+\varepsilon. \tag{2}$$

Combining (1) and (2), for all $k \ge N$,

$$\rho(A) \le \|\!|A^k|\!\|^{1/k} < \rho(A)+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, by the squeeze theorem

$$\rho(A) = \lim_{k\to\infty} \|\!|A^k|\!\|^{1/k}. \qquad \blacksquare$$

Problem 7

Assume $\|\!|I - A|\!\| < 1$ for some matrix norm.

Step 1: A is invertible. Suppose A is not invertible. By the Lemma in Problem 2, $\|\!|I-A|\!\| \ge 1$ — contradicting $\|\!|I-A|\!\| < 1$. (Directly: some $x \ne 0$ has $Ax = 0$, so $(I-A)x = x$ and $\rho(I-A) \ge 1$, while $\rho(I-A) \le \|\!|I-A|\!\| < 1$.) Hence A is non-singular.

Step 2: the inverse. Claim:

$$A^{-1} = \sum_{k=0}^{\infty} (I-A)^k.$$

Since $\|\!|I-A|\!\| < 1$, we have $\sum_{k=0}^{\infty} \|\!|(I-A)^k|\!\| \le \sum_{k=0}^{\infty} \|\!|I-A|\!\|^k < \infty$ (geometric series). By Problem 8 (with $a_k = 1$ and matrix $I-A$), the series converges to some matrix C. Then

$$\begin{aligned} AC &= A \lim_{n\to\infty} \sum_{k=0}^{n} (I-A)^k = \lim_{n\to\infty} \big(I - (I-A)\big) \sum_{k=0}^{n} (I-A)^k \\ &= \lim_{n\to\infty} \Big(\sum_{k=0}^{n} (I-A)^k - \sum_{k=0}^{n} (I-A)^{k+1} \Big) = \lim_{n\to\infty} \big[I - (I-A)^{n+1} \big]. \end{aligned}$$

Since $\|\!|I-A|\!\| < 1$, Lemma 1 gives $(I-A)^{n+1} \to 0$, so $AC = I$. For square matrices this means $C = A^{-1}$:

$$\boxed{A^{-1} = \sum_{k=0}^{\infty} (I-A)^k} \qquad \blacksquare$$

Problem 8

Definitions - For matrices X_k, the series $\sum_{k=0}^{\infty} X_k$ converges to S if the partial sums $S_N = \sum_{k=0}^{N} X_k \to S$. It is *absolutely convergent* if $\sum_{k=0}^{\infty} \|\!|X_k|\!\| < \infty$.

Claim: absolute convergence $\Rightarrow$ convergence. Assume $\sum_k \|\!|X_k|\!\| < \infty$. For $N > M$,

$$\|\!|S_N - S_M|\!\| = \left\|\!\left| \sum_{k=M+1}^{N} X_k \right|\!\right\| \leq \sum_{k=M+1}^{N} \|\!|X_k|\!\| .$$

The right side is a piece of the tail of a convergent numerical series, so it tends to 0 as $M, N \to \infty$. Thus (S_N) is a Cauchy sequence in M_n. Since M_n is a finite-dimensional normed space (hence complete), (S_N) converges. So the series converges.

Our problem. Take $X_k = a_k A^k$. By submultiplicativity,

$$\sum_{k=0}^{\infty} \left\|\!\left| a_k A^k \right|\!\right\| = \sum_{k=0}^{\infty} |a_k| \left\|\!\left| A^k \right|\!\right\| \ \leq \ \sum_{k=0}^{\infty} |a_k| \ \|\!|A|\!\|^k < \infty$$

by hypothesis. Hence $\sum_{k=0}^{\infty} a_k A^k$ is absolutely convergent, and therefore convergent. ■

Problem 9

Let $\|\!|\cdot|\!\|$ be the matrix norm induced by $\|\cdot\|$ and $\|\!|\cdot|\!\|_D$ the matrix norm induced by $\|\cdot\|^D$. We show $\|\!|A|\!\|_D = \left\|\!\left| A^H \right|\!\right\|$.

There are two observations:

(1) For $z \in \mathbb{C}^n$: $\|z\|^D = \max_{\|x\|=1} |z^H x| = \max_{y \neq 0} \dfrac{|y^H z|}{\|y\|}$ (using $|z^H y| = |y^H z|$ and homogeneity).

(2) (Duality theorem) $(\|\cdot\|^D)^D = \|\cdot\|$, i.e. for $w \in \mathbb{C}^n$: $\|w\| = \max_{x \neq 0} \dfrac{|w^H x|}{\|x\|^D}$.

Proof:

$$\begin{aligned} \|\!|A|\!\|_D &= \max_{x \neq 0} \frac{\|Ax\|^D}{\|x\|^D} \overset{(1)}{=} \max_{x \neq 0} \frac{1}{\|x\|^D} \max_{y \neq 0} \frac{|y^H A x|}{\|y\|} = \max_{x \neq 0,\, y \neq 0} \frac{|(A^H y)^H x|}{\|x\|^D \, \|y\|} \\ &= \max_{y \neq 0} \frac{1}{\|y\|} \max_{x \neq 0} \frac{|(A^H y)^H x|}{\|x\|^D} \overset{(2)}{=} \max_{y \neq 0} \frac{\|A^H y\|}{\|y\|} = \left\|\!\left| A^H \right|\!\right\| . \end{aligned}$$

Here we used $y^H A x = (A^H y)^H x$ and interchanged the two maxima. Hence

$$\boxed{\|\!|A|\!\|_D = \left\|\!\left| A^H \right|\!\right\|} \qquad \blacksquare$$

Problem 10

Let $A \in M_n$ be invertible. We show $\min_{x\neq 0} \dfrac{\|Ax\|_2}{\|x\|_2} = \dfrac{1}{\|A^{-1}\|_2}$.

Proof 1 (direct): Since A is invertible, $x \mapsto y = Ax$ is a bijection of $\mathbb{C}^n \setminus \{0\}$, with $x = A^{-1}y$. Therefore

$$\min_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \min_{y\neq 0} \frac{\|y\|_2}{\|A^{-1}y\|_2} = \frac{1}{\max\limits_{y\neq 0} \dfrac{\|A^{-1}y\|_2}{\|y\|_2}} = \frac{1}{\|A^{-1}\|_2}.$$

Proof 2 (via SVD): Let $A = U\Sigma V^H$ with U, V unitary and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_1 \geq \cdots \geq \sigma_n > 0$ ($\sigma_n > 0$ since A is non-singular). Unitary matrices preserve the Euclidean norm, so with $z = V^H x$ (so $\|z\|_2 = \|x\|_2$),

$$\|Ax\|_2^2 = \|\Sigma z\|_2^2 = \sum_{i=1}^{n} \sigma_i^2 |z_i|^2.$$

This lies between $\sigma_n^2 \|z\|_2^2$ and $\sigma_1^2 \|z\|_2^2$, with equality at $z = e_n$ and $z = e_1$ respectively. Hence

$$\|A\|_2 = \sigma_1(A) \ (= \sqrt{\lambda_{\max}(A^H A)}), \qquad \min_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_n(A). \tag{1}$$

Now $A^{-1} = V\Sigma^{-1}U^H$ with $\Sigma^{-1} = \mathrm{diag}(1/\sigma_1, \ldots, 1/\sigma_n)$, which is an SVD of A^{-1} whose largest singular value is $1/\sigma_n$. By (1) applied to A^{-1},

$$\|A^{-1}\|_2 = \frac{1}{\sigma_n(A)}. \tag{2}$$

From (1) and (2): $\min_{x\neq 0} \dfrac{\|Ax\|_2}{\|x\|_2} = \sigma_n(A) = \dfrac{1}{\|A^{-1}\|_2}$. ■

Problem 11

Let A be invertible and B non-invertible. Since B is singular, there is $x \neq 0$ with $Bx = 0$. By definition of the induced norm,

$$\|A - B\|_2 = \max_{z\neq 0} \frac{\|(A-B)z\|_2}{\|z\|_2} \ \geq \ \frac{\|(A-B)x\|_2}{\|x\|_2} = \frac{\|Ax\|_2}{\|x\|_2},$$

because $(A - B)x = Ax$. Using Problem 10,

$$\frac{\|Ax\|_2}{\|x\|_2} \ \geq \ \min_{z\neq 0} \frac{\|Az\|_2}{\|z\|_2} = \frac{1}{\|A^{-1}\|_2}.$$

Therefore

$$\|A - B\|_2 \ \geq \ \frac{1}{\|A^{-1}\|_2}. \qquad \blacksquare$$