PROBLEM SET 2 SOLUTIONS

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E2 212: Matrix Theory

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Problem 1

(a) Proof of subadditivity of rank. For any $X, Y \in \mathbb{C}^{m\times n}$,

$$\operatorname{rank}(X+Y) \le \operatorname{rank}(X) + \operatorname{rank}(Y).$$

For any vector v,

$$(X+Y)v = Xv + Yv \in \mathcal{R}(X) + \mathcal{R}(Y),$$

so

$$\mathcal{R}(X+Y) \subseteq \mathcal{R}(X) + \mathcal{R}(Y).$$

Applying the dimension inequality

$$\dim(U+W) \le \dim(U) + \dim(W),$$

we obtain

$$\begin{aligned}\dim\Big(\mathcal{R}(X+Y)\Big) &\le \dim\Big(\mathcal{R}(X) + \mathcal{R}(Y)\Big) \\ &\le \dim\Big(\mathcal{R}(X)\Big) + \dim\Big(\mathcal{R}(Y)\Big),\end{aligned}$$

and hence

$$\operatorname{rank}(X+Y) \le \operatorname{rank}(X) + \operatorname{rank}(Y).$$

Now, since

$$A = (A+B) + (-B),$$

subadditivity gives

$$\operatorname{rank}(A) \le \operatorname{rank}(A+B) + \operatorname{rank}(-B).$$

Because $\operatorname{rank}(-B) = \operatorname{rank}(B) = k$ and $\operatorname{rank}(A) = r$,

$$r \le \operatorname{rank}(A+B) + k \quad \Longrightarrow \quad \operatorname{rank}(A+B) \ge r - k.$$

Together with

$$\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B) = r + k,$$

we get

$$\boxed{r - k \le \operatorname{rank}(A+B) \le r + k}.$$

(b) Write

$$A = (A-B) + B.$$

Then

$$\operatorname{rank}(A) \le \operatorname{rank}(A-B) + \operatorname{rank}(B),$$

so

$$\text{rank}(A - B) \geq r - k.$$

Similarly,

$$B = A - (A - B) = A + (B - A),$$

which gives

$$\text{rank}(B) \leq \text{rank}(A) + \text{rank}(B - A) = r + \text{rank}(A - B).$$

Therefore,

$$\text{rank}(A - B) \geq k - r = -(r - k).$$

Combining the two inequalities,

$$\boxed{|r - k| \leq \text{rank}(A - B)}.$$

Problem 2

Condition for $\langle x, y\rangle_S = \langle Sx, Sy\rangle$ to be an inner product.

Non-negativity holds because

$$\langle x, x\rangle_S = \langle Sx, Sx\rangle \geq 0.$$

For positive definiteness, we require

$$\langle x, x\rangle_S = 0 \iff x = 0.$$

Since

$$\langle Sx, Sx\rangle = 0 \iff Sx = 0,$$

we need

$$\text{Null}(S) = \{0\}.$$

Therefore, S must be nonsingular (invertible), equivalently it must have full column rank.

$$\boxed{S \text{ must be invertible (nonsingular).}}$$

Problem 3

Cauchy-Schwarz inequality. If $y = 0$, then both sides are zero and equality holds. Now let $y \neq 0$ and define

$$c = \frac{\langle x, y\rangle}{\langle y, y\rangle}.$$

Since the inner product is non-negative,

$$\langle x - cy, x - cy\rangle \geq 0.$$

Expanding,

$$0 \leq \langle x, x\rangle - c\langle y, x\rangle - \overline{c}\langle x, y\rangle + |c|^2\langle y, y\rangle.$$

Substituting $c = \langle x, y\rangle/\langle y, y\rangle$ gives

$$\langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} \geq 0.$$

Hence,

$$\boxed{|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle}.$$

Equality holds if and only if

$$\langle x - cy, x - cy\rangle = 0,$$

which, by positive definiteness, implies

$$x - cy = 0 \quad \Longrightarrow \quad x = cy.$$

Thus x and y are linearly dependent, and conversely linear dependence gives equality.

Problem 4

Define

$$\|x\| = \langle x, x\rangle^{1/2}.$$

We verify the norm properties.

(i) Non-negativity and definiteness.

$$\|x\| = 0 \iff \langle x, x\rangle = 0 \iff x = 0.$$

(ii) Homogeneity.

$$\begin{aligned}\|\alpha x\|^2 &= \langle \alpha x, \alpha x\rangle \\ &= \alpha\overline{\alpha}\langle x, x\rangle \\ &= |\alpha|^2\|x\|^2.\end{aligned}$$

Therefore,

$$\boxed{\|\alpha x\| = |\alpha|\,\|x\|}.$$

(iii) Triangle inequality.

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2.\end{aligned}$$

Using $\mathrm{Re}(z) \le |z|$ and the Cauchy-Schwarz inequality,

$$\begin{aligned}\|x+y\|^2 &\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2.\end{aligned}$$

Taking square roots,

$$\boxed{\|x+y\| \le \|x\| + \|y\|}.$$

Problem 5

For a semi-inner product, positivity is not guaranteed; that is, $\langle x, x\rangle$ can be zero even when $x \neq 0$.

Suppose $\langle y, y\rangle = 0$. If $\langle x, y\rangle \neq 0$, consider

$$\langle x + \lambda y, x + \lambda y\rangle \ge 0.$$

Expanding,

$$\langle x, x\rangle + \lambda\langle y, x\rangle + \overline{\lambda}\langle x, y\rangle + |\lambda|^2\langle y, y\rangle \ge 0.$$

Since $\langle y, y\rangle = 0$,

$$\langle x, x\rangle + 2\,\mathrm{Re}\Big(\overline{\lambda}\langle x, y\rangle\Big) \ge 0.$$

By choosing an appropriate λ, the real part can be made arbitrarily negative, contradicting the inequality. Hence,

$$\langle x, y\rangle = 0.$$

Therefore, in all cases,

$$\boxed{|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle}.$$

The function $\|x\| = \langle x, x\rangle^{1/2}$ satisfies homogeneity and the triangle inequality, but strict positivity may fail. Hence it is a seminorm.

Problem 6

Parallelogram identity.

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \Big(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle\Big) \\ &\quad + \Big(\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle\Big) \\ &= 2\langle x, x\rangle + 2\langle y, y\rangle \\ &= 2(\|x\|^2 + \|y\|^2).\end{aligned}$$

Thus,

$$\boxed{\frac{1}{2}\Big(\|x+y\|^2 + \|x-y\|^2\Big) = \|x\|^2 + \|y\|^2}.$$

Problem 7

Let $p(x)$ be an arbitrary seminorm and define

$$N = \{x \in \mathbb{C}^n : p(x) = 0\}.$$

The set N is a subspace. Construct a complementary subspace W such that

$$\mathbb{C}^n = N \oplus W.$$

Thus, every $x \in \mathbb{C}^n$ has a unique decomposition

$$x = x_N + x_W, \qquad x_N \in N, \quad x_W \in W.$$

Let $S \in M_n$ be the projection onto W along N. Then

$$Sx = x_W, \qquad (I - S)x = x_N.$$

Using the triangle inequality for p, we have

$$\begin{aligned} p(x) &= p(x_N + x_W) \\ &\leq p(x_N) + p(x_W) \\ &= p(x_W). \end{aligned}$$

Also,

$$p(x_W) = p(x - x_N) \leq p(x) + p(-x_N) = p(x),$$

so

$$p(x) = p(x_W) = p(Sx).$$

Now define

$$\|x\| = p(Sx) + \|(I - S)x\|_2,$$

where $\|\cdot\|_2$ is the standard Euclidean norm. This is a norm: it is positive definite because the two terms can vanish simultaneously only when $x \in W \cap N = \{0\}$; homogeneity follows from homogeneity of p and $\|\cdot\|_2$; and the triangle inequality follows from the corresponding properties of the two terms.

Since S is a projection, $S^2 = S$, and hence

$$\begin{aligned} \|Sx\| &= p(S(Sx)) + \|(I - S)Sx\|_2 \\ &= p(Sx) + 0 \\ &= p(x). \end{aligned}$$

Therefore,

$$\boxed{p(x) = \|Sx\| = \|x\|_S}.$$

Problem 8

To prove,

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|.$$

Since $x = y + (x - y)$,

$$\|x\| \leq \|y\| + \|x - y\|,$$

so

$$\|x\| - \|y\| \leq \|x - y\|.$$

Similarly, since $y = x + (y - x)$,

$$\|y\| \leq \|x\| + \|y - x\| = \|x\| + \|x - y\|,$$

which gives

$$\|y\| - \|x\| \leq \|x - y\|.$$

Combining the two inequalities,

$$\boxed{\Big|\|x\| - \|y\|\Big| \leq \|x - y\|}.$$

Problem 9

Minkowski's inequality. Let $p > 1$ and let q satisfy

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$p - 1 = \frac{p}{q}.$$

For each i,

$$\begin{aligned} |x_i + y_i|^p &= |x_i + y_i|\,|x_i + y_i|^{p-1} \\ &\leq (|x_i| + |y_i|)|x_i + y_i|^{p/q} \\ &= |x_i|\,|x_i + y_i|^{p/q} + |y_i|\,|x_i + y_i|^{p/q}. \end{aligned}$$

Summing over i,

$$\sum_i |x_i + y_i|^p \leq \sum_i |x_i|\,|x_i + y_i|^{p/q} + \sum_i |y_i|\,|x_i + y_i|^{p/q}.$$

Applying Hölder's inequality to the first sum,

$$\begin{aligned} \sum_i |x_i|\,|x_i + y_i|^{p/q} &\leq \left(\sum_i |x_i|^p\right)^{1/p} \left(\sum_i \left(|x_i + y_i|^{p/q}\right)^q\right)^{1/q} \\ &= \left(\sum_i |x_i|^p\right)^{1/p} \left(\sum_i |x_i + y_i|^p\right)^{1/q}. \end{aligned}$$

Similarly,

$$\sum_i |y_i|\,|x_i + y_i|^{p/q} \leq \left(\sum_i |y_i|^p\right)^{1/p} \left(\sum_i |x_i + y_i|^p\right)^{1/q}.$$

Therefore,

$$\sum_i |x_i + y_i|^p \leq \left[\left(\sum_i |x_i|^p\right)^{1/p} + \left(\sum_i |y_i|^p\right)^{1/p}\right] \left(\sum_i |x_i + y_i|^p\right)^{1/q}.$$

Assuming $\|x+y\|_p \neq 0$ (otherwise the inequality is trivial), divide by $(\sum_i |x_i+y_i|^p)^{1/q}$. Since $1-1/q = 1/p$,

$$\boxed{\left(\sum_i |x_i+y_i|^p\right)^{1/p} \leq \left(\sum_i |x_i|^p\right)^{1/p} + \left(\sum_i |y_i|^p\right)^{1/p}}.$$

This is Minkowski's inequality.

Problem 10

Show that

$$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p.$$

Recall

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}.$$

Let

$$|x_M| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Then

$$\|x\|_p = |x_M| \left(\frac{|x_1|^p}{|x_M|^p} + \cdots + \frac{|x_n|^p}{|x_M|^p}\right)^{1/p}.$$

As $p \to \infty$, every ratio $|x_i|/|x_M|$ with $|x_i| < |x_M|$ tends to zero after raising to the p-th power, while each ratio with $|x_i| = |x_M|$ contributes 1. If there are k such maximal entries, the parenthesized factor tends to

$$k^{1/p} \longrightarrow 1.$$

Hence,

$$\boxed{\lim_{p\to\infty} \|x\|_p = |x_M| = \|x\|_\infty}.$$

Problem 11

Show that $\|x\|_\infty$ is not derived from an inner product.

If a norm is induced by an inner product, it must satisfy the parallelogram identity. Consider

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{in } \mathbb{R}^2.$$

Then

$$\|x\|_\infty = 1, \qquad \|y\|_\infty = 1.$$

Also,

$$x + y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \|x + y\|_\infty = 1,$$

and

$$x - y = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \|x - y\|_\infty = 1.$$

For an inner-product norm, the parallelogram identity requires

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Here,

$$\text{LHS} = 1^2 + 1^2 = 2,$$

whereas

$$\text{RHS} = 2(1^2 + 1^2) = 4.$$

Since $2 \neq 4$, the identity fails. Therefore,

$$\boxed{\|\cdot\|_\infty \text{ is not induced by an inner product.}}$$

Problem 12

Let

$$\|x\|_S = \|Sx\|,$$

where $S \in M_{m,n}$ has full column rank and $m \geq n$. We verify that $\|\cdot\|_S$ is a norm on $\mathbb{C}^n$.

(a) Non-negativity.

$$\|x\|_S = \|Sx\| \geq 0.$$

(b) Positivity.

$$\|Sx\| = 0 \iff Sx = 0.$$

Since S has full column rank, its columns are linearly independent. Therefore,

$$Sx = \sum_i x_i S_i = 0 \quad \Longrightarrow \quad x_i = 0 \text{ for all } i,$$

so $x = 0$. Hence,

$$\|x\|_S = 0 \iff x = 0.$$

(c) Homogeneity. For any scalar $c \in \mathbb{C}$,

$$\begin{aligned} \|cx\|_S &= \|S(cx)\| \\ &= \|c(Sx)\| \\ &= |c| \, \|Sx\| \\ &= |c| \, \|x\|_S. \end{aligned}$$

(d) Subadditivity. For $x, y \in \mathbb{C}^n$,

$$\begin{aligned} \|x+y\|_S &= \|S(x+y)\| \\ &= \|Sx + Sy\| \\ &\leq \|Sx\| + \|Sy\| \\ &= \|x\|_S + \|y\|_S. \end{aligned}$$

Therefore,

$$\boxed{\|x\|_S = \|Sx\| \text{ is a norm on } \mathbb{C}^n.}$$