E2 212 Matrix Theory
Quiz 1

Time: 20 Minutes **Date: 22/08/2026**

Question 1 3 points

Let

$$V = \left\{A = [a_{ij}] \in \mathbb{R}^{m\times n} : a_{ii} = 0, \quad i = 1, \ldots, n\right\}, \qquad m > n.$$

Determine whether V is a vector space or not. If it is, find a basis and the dimension of V.

Solution:

V is a vector space: $0 \in V$, and if $A, B \in V$, $\alpha \in \mathbb{R}$, then $(A+B)_{ii} = 0$ and $(\alpha A)_{ii} = 0$, so V is a subspace of $\mathbb{R}^{m\times n}$.

Basis: $\{E_{ij} : 1 \le i \le m,\ 1 \le j \le n,\ i \ne j\}$, where E_{ij} has a 1 at (i,j) and 0 elsewhere (the standard basis without $E_{11}, \ldots, E_{nn}$).

$\dim V = mn - n = n(m-1)$.

Question 2 5 points

Let

$$A \in \mathbb{R}^{m\times n}.$$

Suppose

$$x \in \mathcal{R}(A) \cap \mathcal{N}(A^T)$$

and

$$y \in \mathcal{R}(A^T) \cap \mathcal{N}(A).$$

Is the following claim always true? Justify your answer.

$$x = y.$$

Solution:

Take any $z_1 \in \mathcal{R}(A)$, $z_2 \in \mathcal{N}(A^T)$. Then $z_1 = Av$ for some v, and $A^T z_2 = 0$:

$$z_1^T z_2 = (Av)^T z_2 = v^T A^T z_2 = v^T 0 = 0.$$

For x: $x \in \mathcal{R}(A)$ and $x \in \mathcal{N}(A^T)$. Take $z_1 = z_2 = x$:

$$x^T x = \sum_i x_i^2 = 0 \ \Rightarrow \ x = 0.$$

For y: take $z_1 \in \mathcal{R}(A^T)$, $z_2 \in \mathcal{N}(A)$. Then $z_1 = A^T u$, $Az_2 = 0$:

$$z_1^T z_2 = (A^T u)^T z_2 = u^T A z_2 = 0.$$

With $z_1 = z_2 = y$:

$$y^T y = \sum_i y_i^2 = 0 \ \Rightarrow \ y = 0.$$

$$\boxed{x = y = 0 \ \text{ for } m = n}$$

for $m \ne n$: zero vectors of different spaces, so comparision is meaningless.