E2 212 Matrix Theory
Quiz 2

Time: 30 Minutes **Date: 03/09/2026**

Question 1 8 points

Let $A, B \in \mathbb{R}^{m\times n}$ and let $\alpha_1, \alpha_2 \in \mathbb{R}$. Define

$$\langle x, y\rangle = \min\left\{x^T(\alpha_1 A^T A)y,\ x^T(\alpha_2 B^T B)y\right\}, \qquad x, y \in \mathbb{R}^n.$$

Find the necessary and sufficient conditions on A, B, α_1, and α_2 under which $\langle\cdot,\cdot\rangle$ defines an inner product on $\mathbb{R}^n$.

Solution:

$P = \alpha_1 A^T A, \quad Q = \alpha_2 B^T B, \quad p = x^T P y, \quad q = x^T Q y, \quad f(x, y) = \min\{p, q\}.$

1. Non-negativity $f(x, x) \geq 0\ \forall x$.

$$f(x, x) = \min\{\alpha_1(Ax)^T(Ax),\ \alpha_2(Bx)^T(Bx)\} \geq 0$$
$$\iff \alpha_1(Ax)^T(Ax) \geq 0 \text{ and } \alpha_2(Bx)^T(Bx) \geq 0 \quad \forall x.$$

Since $(Ax)^T(Ax) \geq 0$, with > 0 for $Ax \neq 0$:
Condition: ($\alpha_1 \geq 0$ or $A = 0$) and ($\alpha_2 \geq 0$ or $B = 0$).

2. Positivity $f(x, x) = 0 \Rightarrow x = 0$. Given 1,

$$f(x, x) = 0 \iff \alpha_1(Ax)^T(Ax) = 0 \text{ or } \alpha_2(Bx)^T(Bx) = 0$$
$$\iff \alpha_1 = 0 \text{ or } Ax = 0 \text{ or } \alpha_2 = 0 \text{ or } Bx = 0.$$

No $x \neq 0$ may satisfy this: $\alpha_1 \neq 0$, $\mathcal{N}(A) = \{0\}$, $\alpha_2 \neq 0$, $\mathcal{N}(B) = \{0\}$.
With 1 ($A, B \neq 0$ as $n \geq 1$): Condition: $\alpha_1, \alpha_2 > 0$, $\operatorname{rank} A = \operatorname{rank} B = n$.

3. Homogeneity $f(cx, y) = c\, f(x, y)$.

$c \geq 0$: $\min\{cp, cq\} = c\min\{p, q\}$ ✓
$c < 0$: $f(cx, y) = \min\{cp, cq\} = c\max\{p, q\}$ (multiplying by $c < 0$ flips order)
required: $c\max\{p, q\} = c\ \min\{p, q\} \iff \max\{p, q\} = \min\{p, q\}$
$\Rightarrow p = q\ \forall x, y \Rightarrow x^T(P - Q)y = 0\ \forall x, y \Rightarrow P = Q.$

Condition: $\alpha_1 A^T A = \alpha_2 B^T B$.

4. Additivity $f(x + z, y) = f(x, y) + f(z, y)$.

$z = -x$: $f(x, y) + f(-x, y) = f(0, y) = 0$
$\Rightarrow \min\{-p, -q\} = -\min\{p, q\} \Rightarrow \max\{p, q\} = \min\{p, q\} \Rightarrow P = Q.$

Conversely $P = Q \Rightarrow f = x^T P y$, additive. Condition: $\alpha_1 A^T A = \alpha_2 B^T B$.

$$\boxed{\alpha_1 A^T A = \alpha_2 B^T B, \quad \alpha_1, \alpha_2 > 0, \quad \operatorname{rank} A = \operatorname{rank} B = n}$$

Question 2 8 points

Let A and B be linear transformations. For $\alpha, \beta \in \mathbb{R}$, define

$$N(x) = \max\left\{\alpha\|Ax\|_a,\ \beta\|Bx\|_b\right\}.$$

(a) Prove or disprove: $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbb{R}^n$. If so, under what conditions?

(b) Prove or disprove: $N(x)$ defines a norm on $\mathbb{R}^n$. If so, state the necessary and sufficient conditions on α, β, A, and B under which N is a norm, and prove your claim.

(c) If the conditions in part (b) fail, is $N(x)$ still guaranteed to be a *semi-norm* on $\mathbb{R}^n$? State the precise conditions on α, β, A, and B under which N is a semi-norm, and justify your answer.

Solution:

$a(x) = \|Ax\|_a \geq 0, \qquad b(x) = \|Bx\|_b \geq 0.$

(a)

$$\begin{aligned} a(cx) &= |c|\, a(x), \\ a(x+y) &= \|Ax + Ay\|_a \leq a(x) + a(y), \\ a(x) = 0 &\iff Ax = 0 \iff x \in \mathcal{N}(A). \end{aligned}$$

Always a semi-norm. Norm $\iff \mathcal{N}(A) = \{0\} \iff \operatorname{rank} A = n$. Same for b with B.

(b) $N(x) = \max\{\alpha a(x), \beta b(x)\}$. Check each property.

(i) Non-negativity $N(x) \geq 0\ \forall x$. Assume $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbb{R}^n$.

$$\begin{aligned} \alpha \geq 0 \text{ or } \beta \geq 0: &\ N(x) \geq 0 \\ \alpha, \beta < 0: &\ N(x) < 0 \end{aligned}$$

Condition: $\max\{\alpha, \beta\} \geq 0$ or $A = 0$ or $B = 0$.

(ii) Positivity $N(x) = 0 \Rightarrow x = 0$. Zero set $Z = \{x : N(x) = 0\}$:

$$\begin{aligned} &\alpha, \beta > 0: && N = 0 \iff a = b = 0 \Rightarrow Z = \mathcal{N}(A) \cap \mathcal{N}(B) \\ &\alpha > 0 \geq \beta: && N = \alpha a \Rightarrow Z = \mathcal{N}(A) \\ &\beta > 0 \geq \alpha: && N = \beta b \Rightarrow Z = \mathcal{N}(B) \\ &\alpha, \beta < 0: && N = 0 \iff \alpha a = 0 \text{ or } \beta b = 0 \Rightarrow Z = \mathcal{N}(A) \cup \mathcal{N}(B) \\ &\alpha = 0 \text{ or } \beta = 0, \text{ other} \leq 0: && N \equiv 0 \Rightarrow Z = \mathbb{R}^n \end{aligned}$$

Condition: $Z = \{0\}$:

$$\begin{aligned} \alpha, \beta > 0: &\ \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\} \\ \alpha > 0 \geq \beta: &\ \mathcal{N}(A) = \{0\} \\ \beta > 0 \geq \alpha: &\ \mathcal{N}(B) = \{0\} \\ \alpha, \beta < 0: &\ \mathcal{N}(A) = \mathcal{N}(B) = \{0\} \end{aligned}$$

(iii) Homogeneity.

$$N(cx) = \max\{\alpha|c|a(x),\ \beta|c|b(x)\} = |c|\, N(x) \qquad \forall\, \alpha, \beta, A, B.$$

(iv) Additivity (triangle) $N(x+y) \leq N(x) + N(y)$.

$$\begin{aligned} \alpha, \beta \geq 0: \ N(x+y) &\leq \max\{\alpha a(x) + \alpha a(y),\ \beta b(x) + \beta b(y)\} \\ &\leq \max\{\alpha a(x), \beta b(x)\} + \max\{\alpha a(y), \beta b(y)\} = N(x) + N(y) \end{aligned}$$

Combine (i)–(iv):

$$\begin{aligned}
\alpha,\beta>0:&\ \text{(i),(iii),(iv) hold;}\ \ \text{(ii)} \Rightarrow \mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}\\
\alpha>0\ge\beta:&\ \text{(i),(iii),(iv) hold;}\ \ \text{(ii)} \Rightarrow \mathcal{N}(A)=\{0\}\\
\beta>0\ge\alpha:&\ \text{(i),(iii),(iv) hold;}\ \ \text{(ii)} \Rightarrow \mathcal{N}(B)=\{0\}\\
\alpha,\beta<0:&\ \text{(i)} \Rightarrow A=0 \text{ or } B=0\\
&\ \text{(ii)} \Rightarrow \mathcal{N}(A)=\mathcal{N}(B)=\{0\} \ \Rightarrow\ \text{contradiction}\\
\alpha=0 \text{ or } \beta=0,\ \text{other}\le 0:&\ \text{(ii) fails}
\end{aligned}$$

$\alpha,\beta>0:$	$\mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}\ \ (\iff \operatorname{rank}\begin{bmatrix}A\\B\end{bmatrix}=n)$
$\alpha>0\ge\beta:$	$\mathcal{N}(A)=\{0\}$
$\beta>0\ge\alpha:$	$\mathcal{N}(B)=\{0\}$
$\alpha,\beta\le 0:$	never

(c) Semi-norm = (i) non-negativity + (iii) homogeneity + (iv) triangle inequality. Positivity (ii) is *not* required, so conditions on $\mathcal{N}(A),\mathcal{N}(B)$ are irrelevant.

(iii) holds always. By (i) and (iv), case by case:

Case	N	Semi-norm?
$\alpha,\beta\ge 0$	$\max\{\alpha a,\beta b\}$	yes, for all A,B
$\alpha\ge 0>\beta$	αa	yes, for all A,B
$\beta\ge 0>\alpha$	βb	yes, for all A,B
$\alpha,\beta<0$	$-\min\{\lvert\alpha\rvert a,\ \lvert\beta\rvert b\}\ \le 0$	only if $N\equiv 0$

Last case: $N\le 0$, so non-negativity forces $N\equiv 0$:

$$\begin{aligned}
N\equiv 0 &\iff \min\{|\alpha|a(x),|\beta|b(x)\}=0\ \ \forall x\\
&\iff a(x)=0 \text{ or } b(x)=0\ \ \forall x\\
&\iff \mathbb{R}^n=\mathcal{N}(A)\cup\mathcal{N}(B)\\
&\iff \mathcal{N}(A)=\mathbb{R}^n \text{ or } \mathcal{N}(B)=\mathbb{R}^n\\
&\quad\text{(a vector space is not a union of two proper subspaces)}\\
&\iff A=0 \text{ or } B=0.
\end{aligned}$$

Failure: $\alpha,\beta<0$, $A\neq 0$, $B\neq 0 \Rightarrow \exists x\notin\mathcal{N}(A)\cup\mathcal{N}(B)$, so $a(x),b(x)>0$ and $N(x)=\max\{\alpha a(x),\beta b(x)\}<0$.

$$N \text{ is a semi-norm} \iff \max\{\alpha,\beta\}\ge 0 \ \text{ or } \ A=0 \ \text{ or } \ B=0$$

So N is *not* guaranteed to be a semi-norm: it fails exactly when $\alpha<0$, $\beta<0$, $A\neq 0$, $B\neq 0$.