E2 212 Matrix Theory
Quiz 3

Time: 20 Minutes **Date: 01/10/2026**

Question 1 3 points

Let $\|\cdot\|$ be any norm on $\mathbb{C}^n$ with dual norm $\|\cdot\|^D$ defined by

$$\|v\|^D = \max_{\|x\|=1} |v^H x|.$$

For $u, v \in \mathbb{C}^n$, show that the induced matrix norm of the rank-one matrix $A = uv^H$ is

$$|||A||| = \|u\| \, \|v\|^D.$$

Solution:

By definition of the induced norm,

$$|||A||| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \|(uv^H)x\| = \max_{\|x\|=1} \|(v^H x)\, u\|.$$

Since $v^H x$ is a scalar, homogeneity of the vector norm gives

$$\|(v^H x)\, u\| = |v^H x| \cdot \|u\|.$$

Hence

$$|||A||| = \|u\| \cdot \max_{\|x\|=1} |v^H x| = \|u\| \, \|v\|^D,$$

using the definition of the dual norm $\|v\|^D$.

Question 2 5 points

Let $A \in M_n$, and $\|\cdot\|_2$ is the norm induced by the Euclidean norm.

(a) Show that

$$|a_{ii}| \le \|A\|_2 \quad \text{for every } i = 1, \ldots, n.$$

(Hint:- think of a function f such that $a_{i,j} = f(A)$.)

(b) When does equality hold in the above inequality?

Solution:

(a) Let e_i denote the i-th standard basis vector, so $\|e_i\|_2 = 1$. Note that

$$a_{ii} = e_i^H A e_i.$$

By the Cauchy–Schwarz inequality,

$$|a_{ii}| = |e_i^H (Ae_i)| \le \|e_i\|_2 \cdot \|Ae_i\|_2 = \|Ae_i\|_2.$$

Since $\|\cdot\|_2$ is the norm induced by the Euclidean norm, property (b) of induced norms gives

$$\|Ae_i\|_2 \le \|A\|_2 \, \|e_i\|_2 = \|A\|_2.$$

Combining the two inequalities:

$$|a_{ii}| \le \|A\|_2.$$

(b) Equality holds for every $i = 1, \ldots, n$ if and only if A is diagonal with

$$|a_{11}| = |a_{22}| = \cdots = |a_{nn}| = \|A\|_2 = \sigma_1(A).$$

Indeed, equality at index i forces $Ae_i = a_{ii} e_i$ and $A^H e_i = \overline{a_{ii}}\, e_i$, so row i and column i vanish off the diagonal. Holding for all i, A is diagonal, and its singular values are $|a_{ii}|$, so all $|a_{ii}|$ equal $\sigma_1(A)$. Equivalently, $A = \sigma_1 D$ with $D = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ unitary.