Minimum variance unbiased estimator (MVU)

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Data model:

$$x[n] = A + w[n] \qquad n = 0, 1, \ldots, N-1$$

$w[n] \sim \mathcal{N}(0, \sigma^2)$

A : deterministic and unknown

$x \sim \mathcal{N}(A, \sigma^2)$

true A = 1

Estimator is a r.v. as well.

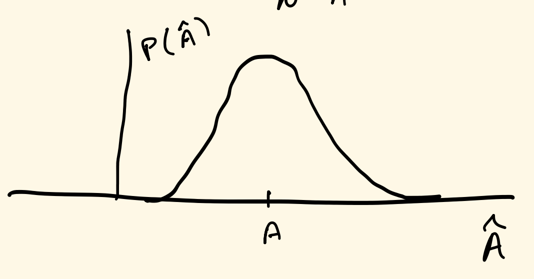
$$\left\{ \begin{array}{l} \hat{A}_1 = x[0] = 0.97 \\ \hat{A}_2 = \frac{1}{N} \sum_{n=0}^{N-1} x[n] = 0.9 \end{array} \right\} \text{ functions of random variable.}$$

Mean of $\hat{A}_1$ and $\hat{A}_2$:

$E[\hat{A}_1] = E[x[0]] = A$

$E[\hat{A}_2] = E\left[\frac{1}{N} \sum_n x[n]\right] = A$

On average both these estimators attain the true value of the unknown parameter



θ : deterministic parameter (unknown)

$\hat{\theta}$: Estimate of θ

$E[\hat{\theta}] = \theta \quad \forall \theta$ "unbiased estimators"

$\underline{x}$: observations

$\underline{x} \sim p(\underline{x}; \theta)$: data pdf.

$\rightarrow \hat{\theta} = g(\underline{x})$

$$E(\hat{\theta}) = \int g(\underline{x})\, p(\underline{x}; \theta)\, d\underline{x} \qquad \forall \theta$$
$$= \theta$$

Having an unbiased estimator does not neccessarily mean that the estimator is good, but having bias will alway result in a poor estimator.

$$\hat{A}_3 = \frac{1}{2N} \sum_{n=0}^{N-1} x[n]$$

$$E[\hat{A}_3] = \frac{1}{2} A \left\{ \begin{array}{ll} \neq A & \forall A \neq 0 \\ = A & A = 0 \end{array} \right\} \text{ biased}$$

$\mathrm{var}[\hat{A}_3] = \frac{\sigma^2}{4N}$. As $N \rightarrow \infty$, $\mathrm{var}[\hat{A}_3] \rightarrow 0$

but $E[\hat{A}_3] \nrightarrow A$

Suppose we have multiple estimates of the same parameter

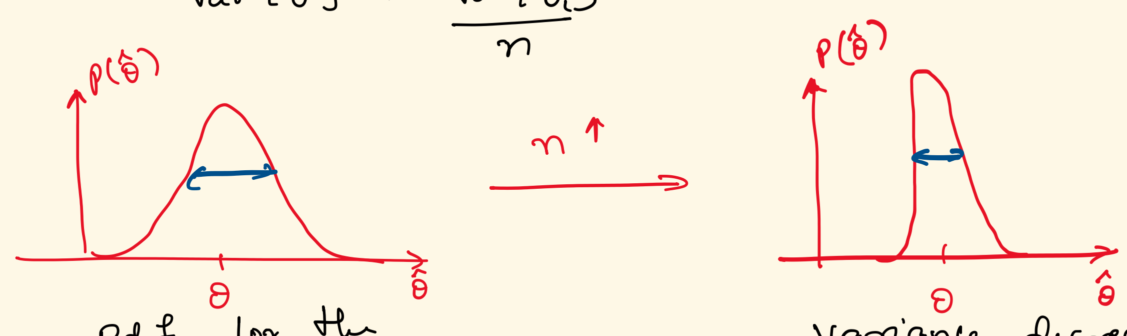
$\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n\}$ ←

$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_i$: reasonable procedure to arrive at a better one

$\{\hat{\theta}_i\}$ are unbiased: (same variance & uncorrelated)

$$E[\hat{\theta}] = E\left[\frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_i\right]$$
$$= \frac{1}{n} \sum_{i=1}^{n} E(\hat{\theta}_i) = \theta \quad \text{(unbiased)}$$

$$\mathrm{var}[\hat{\theta}] = \frac{\mathrm{var}[\hat{\theta}_i]}{n}$$



pdf for the combined estimator

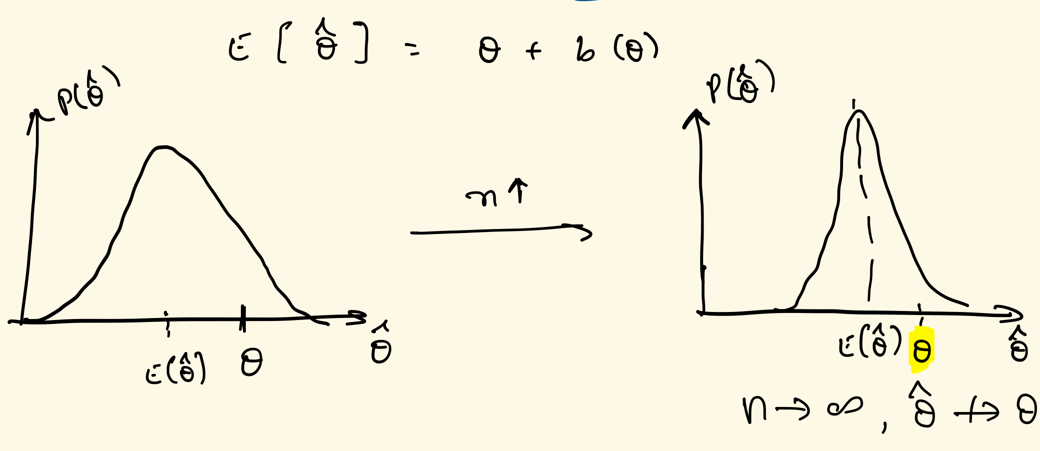
Variance decreases $\hat{\theta} \rightarrow \theta$

$\{\hat{\theta}_i\}$ are biased:

$b(\theta)$ as "bias"

$E[\hat{\theta}_i] = \theta + b(\theta) \quad : b(\theta) = E[\hat{\theta}_i] - \theta$

$E[\hat{\theta}] = \theta + b(\theta)$



$n \rightarrow \infty$, $\hat{\theta} \nrightarrow \theta$

Minimum variance:

Mean Square Error: (MSE) $E\{(\theta - \hat{\theta})^2\} = \mathrm{mse}(\hat{\theta})$ "optimality criterion"

$$\mathrm{mse}(\hat{\theta}) = E\{[(\hat{\theta} - E(\hat{\theta})) + (E(\hat{\theta}) - \theta)]^2\}$$
$$= E[(\hat{\theta} - E(\hat{\theta}))^2] + (E(\hat{\theta}) - \theta)^2$$
$$+ \underbrace{2E[(\hat{\theta} - E(\hat{\theta}))(E(\hat{\theta}) - \theta)]}_{0}$$
$$= \mathrm{Var}(\hat{\theta}) + b^2(\theta)$$

→ "not realizable"

Example:

$$\check{A} = \frac{a}{N} \sum_{n=0}^{N-1} x[n]$$

find "a" that minimizes the mse

$E[\check{A}] = aA \qquad \mathrm{var}(\check{A}) = \frac{a^2 \sigma^2}{N}; \quad b(A) = aA - A$

$$\mathrm{mse}(\check{A}) = \frac{a^2 \sigma^2}{N} + (a-1)^2 A^2$$

$\min_a \mathrm{mse}(\check{A})$

$$\frac{d\, \mathrm{mse}(\check{A})}{da} = 0 \Rightarrow 2a\frac{\sigma^2}{N} + 2(a-1)A^2 = 0$$

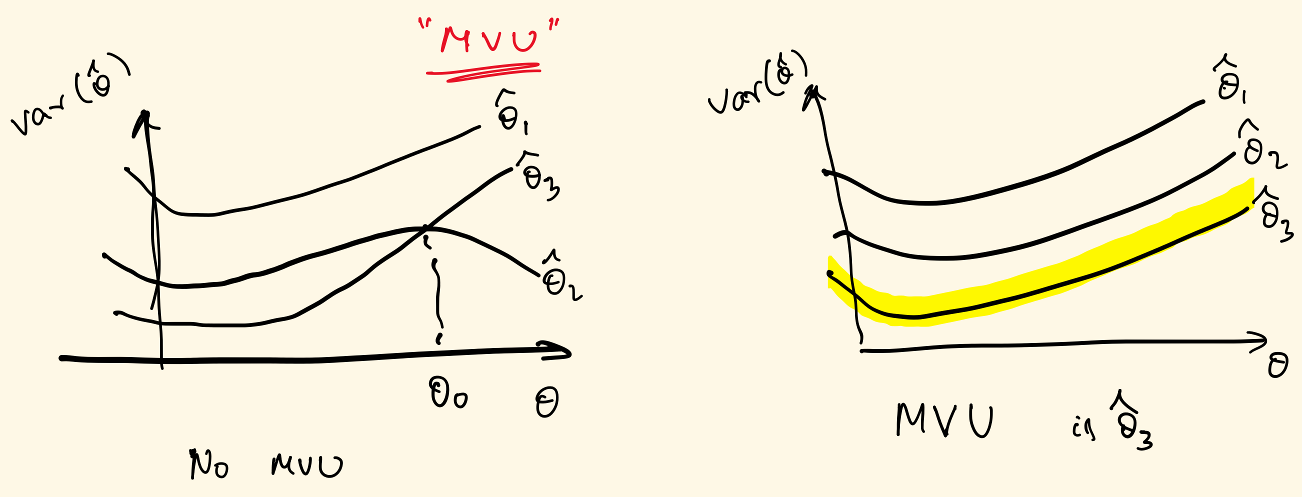
$$\frac{d^2 \mathrm{mse}(\check{A})}{da^2} = \frac{2\sigma^2}{N} + 2A^2 \geq 0$$

$$a_{opt} = \frac{A^2}{A^2 + \frac{\sigma^2}{N}} \quad \leftarrow \text{depends on the unknown 'A'}$$

$$\check{A} = \frac{a_{opt}}{N} \sum_{n=0}^{N-1} x[n] \quad \leftarrow \text{Not realizable}$$

We constrain $b(\theta)$ to zero and seek estimators that minimize the variance.

"MVU"



No MVU

MVU is $\hat{\theta}_3$

- MVU need not always exist. ←
- Recipe to find MVU?

Example:

$x[0] \sim \mathcal{N}(\theta, 1)$

$$x[1] \sim \left\{ \begin{array}{ll} \mathcal{N}(\theta, 1) & \text{if } \theta \geq 0 \\ \mathcal{N}(\theta, 2) & \theta < 0 \end{array} \right.$$

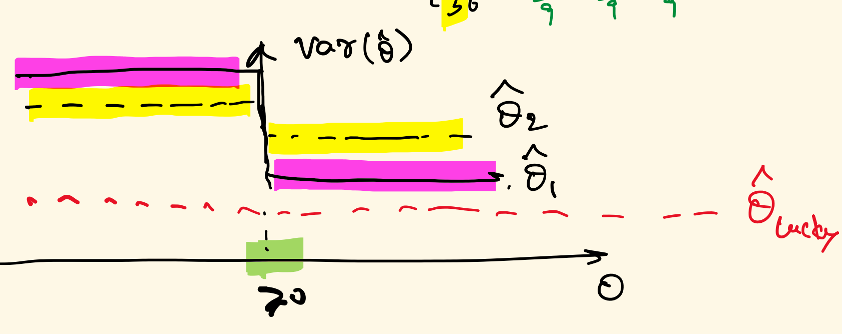
$\hat{\theta}_1 = \frac{1}{2}[x[0] + x[1]] \qquad \hat{\theta}_2 = \frac{2}{3} x[0] + \frac{1}{3} x[1]$

$\mathrm{var}(\hat{\theta}_1) = \frac{1}{4}[\mathrm{var}(x[0]) + \mathrm{var}(x[1])]$

$\mathrm{var}(\hat{\theta}_2) = \frac{4}{9} \mathrm{var}(x[0]) + \frac{1}{9} \mathrm{var}(x[1])$

$$\mathrm{var}(\hat{\theta}_1) = \left\{ \begin{array}{ll} 18/36 \Leftarrow \frac{1}{4} + \frac{1}{4} = \frac{2}{4} & \theta \geq 0 \\ 27/36 = \frac{1}{4} + \frac{2}{4} = \frac{3}{4} & \theta < 0 \end{array} \right.$$

$$\mathrm{var}(\hat{\theta}_2) = \left\{ \begin{array}{ll} 20/36 = \frac{4}{9} + \frac{1}{9} = \frac{5}{9} & \theta \geq 0 \\ 24/36 = \frac{4}{9} + \frac{2}{9} = \frac{6}{9} & \theta < 0 \end{array} \right.$$



(no standard recipe to find such an estimator)

- Cramér-Rao Lower bound (CRLB)
- Rao-Blackwell-Lehmann-Scheffe theorem (Sufficient statistic)
- BLUE (Linear estimator)