Generalised MVU contd.

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Neyman - Fisher factorization:

Proof:

$$p(\underline{x};\theta) = g(T(\underline{x}),\theta)\, h(\underline{x})$$

$T(x)$: functionally dependent on x

① 2 random variables x and y such that $x = y$

$$p(x,y) = p(x)\,\delta(x-y)$$

② Transformation of random variable

$$y = g(\underline{x})$$

$$p(y) = \int p(\underline{x})\,\delta(y - g(x))\,d\underline{x}$$

joint PDF

$$p(\underline{x}, T(\underline{x});\theta)$$

This joint PDF must be zero when $\underline{x} = \underline{x}_0$

$\Rightarrow T(\underline{x}) = T_0$ unless $T(x_0) = T_0$

① Assume that the factorization holds:

$$p(\underline{x} \mid T(\underline{x}) = T_0;\theta) = \frac{p(\underline{x}, T(x) = T_0;\theta)}{p(T(x) = T_0;\theta)}$$

$$= \frac{p(\underline{x};\theta)\,\delta(T(\underline{x}) - T_0)}{p(T(x) = T_0;\theta)}$$

$$= \frac{g(T(x) = T_0;\theta)\, h(x)\,\delta(T(x) - T_0)}{p(T(x) = T_0;\theta)}$$

$$p(T(x) = T_0;\theta) = \int p(\underline{x};\theta)\,\delta(T(\underline{x}) - T_0)\,d\underline{x}$$

$$= \int g(T(x) = T_0;\theta)\, h(x)\,\delta(T(\underline{x}) - T_0)\,d\underline{x}$$

$$= g(T(x) - T_0;\theta)\int h(x)\,\delta(T(x) - T_0)\,d\underline{x}$$

$$p(\underline{x} \mid T(x) = T_0;\theta) = \frac{h(x)\,\delta(T(x) - T_0)}{\int h(x)\,\delta(T(x) - T_0)\,dx}$$

This is independent of θ. Thus, $T(x)$ is a sufficient statistic.

② if $T(x)$ is a sufficient statistic, then the factorization holds.

$$p(\underline{x}, T(x) = T_0;\theta) = \underbrace{p(\underline{x} \mid T(x) = T_0;\theta)}\, p(T(x) = T_0;\theta)$$

$$= p(\underline{x} \mid T(x) = T_0)\, p(T(x) = T_0;\theta)$$

$$p(\underline{x} \mid T(x) = T_0) = w(x)\,\delta(T(x) - T_0)$$

Such that $\underbrace{\int w(x)\,\delta(T(x) - T_0)\,d\underline{x}} = 1$

$$p(\underline{x};\theta)\,\delta(T(x) - T_0) = w(x)\,\delta(T(x) - T_0)\, p(T(x) = T_0;\theta)$$

use $$w(\underline{x}) = \frac{h(x)}{\int h(x)\,\delta(T(x) - T_0)\,d\underline{x}}$$

$$p(\underline{x};\theta)\,\delta(T(x) - T_0) = \frac{h(x)\,\delta(T(\underline{x}) - T_0)}{\int h(x)\,\delta(T(x) - T_0)\,d\underline{x}}\, p(T(x) = T_0;\theta)$$

$$p(\underline{x};\theta) = g(T(x) = T_0;\theta)\, h(x)$$

where

$$g(T(\underline{x}) = T_0;\theta) = \frac{p(T(x) = T_0;\theta)}{\int h(\underline{x})\,\delta(T(x) - T_0)\,d\underline{x}}$$

$$p(T(x) = T_0;\theta) = g(T(x) = T_0;\theta)\int h(x)\,\delta(T(x) - T_0)\,d\underline{x}$$

$T(x) \longrightarrow \hat{\theta}$

Rao - Blackwell - Lehmann - Scheffe:

$\check{\theta}$ is an unbiased estimator of θ and $T(x)$ is a sufficient static for θ

then $\hat{\theta} = E[\check{\theta} \mid T(x)]$

① valid estimator (doesn't depend on θ)

② $\hat{\theta}$ is unbiased

③ $\mathrm{var}(\hat{\theta}) \leq \mathrm{var}(\check{\theta})$

Finally, if $T(x)$ is complete then $\hat{\theta}$ is the MVU estimator.

Alternatively, find some function $g(T(x))$ that is unbiased.

$$T(x) = \sum_{n=0}^{N-1} x[n] \qquad E(T(x)) = NA$$

$$E(g(T(x))) = A \qquad g(\cdot) = \frac{1}{N}$$

$T(x)$ is complete:

$\Longrightarrow$ $E(v(T)) = 0 \quad \forall\,\theta$ $\Longleftarrow$

implies $v(T) = 0$

$$E(g_1(T(\underline{x}))) = \theta \qquad \forall\theta$$

$$E(g_2(T(\underline{x}))) = \theta$$

$$E[\underbrace{g_1(T(x)) - g_2(T(x))}] = 0 \qquad \forall\theta$$

$$g_1 = g_2.$$

Proof:

$\check{\theta}$ is an unbiased estimator

$E(\check{\theta}) = \theta$

① $$\hat{\theta} = E(\check{\theta} \mid T(x))$$

$$= \int \check{\theta}(x)\, p(\underline{x} \mid T(x); \theta)\,d\underline{x}$$

(with "$;\theta$" struck out)

$\longrightarrow$ $$= \int \check{\theta}(x)\, p(\underline{x} \mid T(x))\,d\underline{x}$$

only be a function of T after integrating $\underline{x}$

② $\hat{\theta}$ is unbiased:

$$E(\hat{\theta}) = \int\int \check{\theta}(x)\, p(\underline{x} \mid T(\underline{x});\theta)\,d\underline{x}\; p(T(\underline{x});\theta)\,dT$$

$$= \int \check{\theta}(x)\int p(\underline{x} \mid T(\underline{x});\theta)\, p(T(\underline{x});\theta)\,dT\,d\underline{x}$$

$$= \int \check{\theta}(\underline{x})\, p(\underline{x};\theta)\,d\underline{x}$$

$$= \underline{E(\check{\theta})} = \theta$$

③ $\mathrm{var}(\check{\theta}) \geq \mathrm{var}(\hat{\theta})$

$\quad x[0] \qquad \bar{x}$

$\quad \sigma^2 \qquad \frac{\sigma^2}{N}$

$$\mathrm{var}(\check{\theta}) = E\left((\check{\theta} - E(\check{\theta}))^2\right)$$

$$= E(\underline{\check{\theta} - \hat{\theta}} + \underline{\hat{\theta} - \theta})^2$$

$$= E(\check{\theta} - \hat{\theta})^2 + E(\hat{\theta} - \theta)^2 + 2\underbrace{E(\check{\theta} - \hat{\theta})(\hat{\theta} - \theta)}_{=0}$$

$$E_{T,\check{\theta}}(\check{\theta} - \hat{\theta})(\hat{\theta} - \theta) = E_T\, E_{\check{\theta}|T}[(\check{\theta} - \hat{\theta})(\hat{\theta} - \theta)]$$

$$E_{\check{\theta}|T}[(\check{\theta} - \hat{\theta})(\hat{\theta} - \theta)]$$

$$= E_{\check{\theta}|T}[\check{\theta} - \hat{\theta}]\cdot(\hat{\theta} - \theta)$$

$$= (\hat{\theta} - \theta)[E_{\check{\theta}|T}(\check{\theta}) - \hat{\theta}]$$

$$= (\hat{\theta} - \theta)(\hat{\theta} - \hat{\theta}) = 0$$

$$\Rightarrow \mathrm{var}(\check{\theta}) = E[(\check{\theta} - \hat{\theta})^2] + \mathrm{var}(\hat{\theta})$$

Example:

$$x[n] = A + w[n]$$

$$T(\underline{x}) = \sum_{n=0}^{N-1} x[n] \sim \mathcal{N}(NA, N\sigma^2)$$

$$E(g(T(\underline{x}))) = A \qquad \forall A$$

$$E(h(T(\underline{x}))) = A$$

$$E[\underbrace{g(T) - h(T)}_{v(T)}] = 0$$

$$\int_{-\infty}^{\infty} v(T)\,\frac{1}{\sqrt{2\pi N\sigma^2}}\exp\left(-\frac{1}{2N\sigma^2}(T - NA)^2\right)dT = 0 \qquad \forall A$$

$$\tau = \frac{T}{N} \qquad v(N\tau) = v'(\tau)$$

$$\int_{-\infty}^{\infty} v'(\tau)\,\frac{N}{\sqrt{2\pi N\sigma^2}}\exp\left(-\frac{N}{2\sigma^2}(A - \tau)^2\right)d\tau = 0 \qquad \forall A$$

convolution $\longrightarrow$ pointwise multiplication in freq. domain.

$$V'(f)\,\underbrace{W(f)}_{\text{Gaussian}} = 0 \qquad \forall f$$

$$\Rightarrow g = h.$$

$p(\underline{x};\theta)$ $\xrightarrow[\text{factorization}]{\text{NF}}$ $T(x)$ (Complete)

$T(x)$ (Complete) → $E(\check{\theta} \mid T(x))$ (tedious) or $g(\;)$ (Eye ball)

$\rightsquigarrow$ $\hat{\theta}$ = MVU