Deptartment of Electrical Communication Engineering
Indian Institute of Science

E1 244 Detection and Estimation

22 June 2020, 09:00am– 23 June 2020, 10am, Final assessment

This assessment is for 20 points.

This final assessment has a turn-in time of 24 hours. This deadline is very strict. Therefore, do not wait till the last minute to upload your answer sheet. LATE SUBMISSIONS WILL GET 0 FOR THIS FINAL ASSESSMENT.

The scanned version of the answer sheets should be uploaded in Microsoft Teams against your name. Do not forget to write your name, student enrollment number in the answer sheet. Make sure the lighting conditions are good and the content is not cropped in case you are taking pictures of the answer sheet. UPLOAD ONLY ONE PDF FILE.

Policy: By submitting the answer sheet, you will be "signing by default" that you have not received or offered any help to another student in answering this homework. In case of any violation of this policy, a FAIL grade will be awarded for the course.

Question 1 (10 points)

(2.5 pts) (a) A sensor is placed in an overhead tank to check if the water level is above or below a certain value. The sensor reading x is a Bernoulli random variable, which reads a value of 1 with probability 3/4. However the sensor reading can be observed only through a *fading channel*. That is, we observe

$$y = hx + w,$$

where h is the fading factor and w is the additive noise. Both h and w are exponential random variables with PDFs

$$p(h) = \begin{cases} e^{-h} & h \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad p(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Assume that h, x, and w are all mutually independent.

Given the observation y, we must guess whether the sensor reading was 0 or 1 so that we can switch on the water pump to start filling the tank. Use a binary hypothesis test to determine the decision rule that minimizes P_e. For this optimum rule, give the expression for P_e.

(2.5 pts) (b) We record 15 samples $\{y[0], y[1], \cdots, y[14]\}$ at the output of a harmonic oscillator, where $y[n] = x[n] + w[n]$ for $n = 0, 1, \ldots, 14$. The sequences $x[n]$ and $w[n]$ represent the signal (i.e., the noise-free output of the oscillator) and noise processes, and are collected in vectors $\mathbf{x} = [x[0], x[1], \cdots, x[14]]^{\mathrm{T}}$ and $\mathbf{w} = [w[0], w[1], \cdots, w[14]]^{\mathrm{T}}$.

Assume that $E(\mathbf{x}) = \mathbf{0}$ and the (i, j)th element of the correlation matrix $\mathbf{R}_x = E(\mathbf{x}\mathbf{x}^{\mathrm{T}})$ is given by $R_x(i, j) = r_{|i-j|}$. Also, assume that $E(\mathbf{w}) = \mathbf{0}$ with $\mathbf{R}_w = 0.1\mathbf{I}$, and that $\mathbf{x}$ and $\mathbf{w}$ are mutually independent.

Use n elements of $y[n]$ to form an LMMSE estimate of $x[15]$ and compute the mean squared error. Also, sketch the mean squared error as a function of the number of observations n for

$$r_{|i-j|} = \cos(0.1\pi|i-j|) \quad \text{and} \quad r_{i-j} = \cos(0.5\pi|i-j|).$$

Which of the correlation structures yield better estimates and why?

(2.5 pts) (c) Suppose $x \in \{0, 1\}$ is a random variable given by

$$x = \begin{cases} 1 & \mathbf{h}^{\mathrm{T}}\boldsymbol{\theta} + w \leq 0 \\ 0 & \mathbf{h}^{\mathrm{T}}\boldsymbol{\theta} + w > 0 \end{cases}$$

where $\boldsymbol{\theta} \in \mathbb{R}^N$ is a vector of latent variables, and w is a zero mean unit variance Gaussian variable.

Given data consisting of pairs $(\mathbf{h}_i, x_i), i = 1, \ldots, M$, formulate the maximum likelihood estimation problem for estimating $\boldsymbol{\theta}$.

(2.5 pts) (d) Consider the binary hypothesis problem

$$\begin{aligned} \mathcal{H}_0 :& \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_0, \mathbf{R}) \\ \mathcal{H}_1 :& \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_1, \mathbf{R}) \end{aligned}$$

where $\boldsymbol{\mu}_i$ is the mean vector under hypothesis $\mathcal{H}_i$ and $\mathbf{R}$ is the covariance matrix of $\mathbf{x}$ common to both the hypothesis. Find the decision rule (i.e., the threshold) for which the probability of false alarm P_{FA} will be equal to the probability of miss detection $1 - P_D$.

Question 2 (10 points)

This question has 4 subquestions divided into two parts on estimation and detection.
Estimation: Two sensors measure DC along with a DC offset as

$$x[n] = (-1)^n a + b + w[n], \quad n = 0, 1, 2, \ldots, N-1.$$

We want to estimate both the DC denoted by a and the DC offset denoted by b. The noise $\mathbf{w} = [w[0], w[1], \cdots, w[N-1]]^T$ is Gaussian distributed as $\mathbf{w} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$.

(2.5 pts) (a) Give the update equations for the sequential (or recursive) least squares estimator of $\boldsymbol{\theta} = [a, b]^T$.

(2.5 pts) (b) Give the explicit expressions for the estimators $\hat{a}$ and $\hat{b}$. Illustrate through a sketch the geometric viewpoint of your least squares estimator for $N = 2$.

Detection: Now we proceed to the detection problem of detecting DC plus DC offset in noise. Consider the binary hypothesis testing problem

$$\begin{aligned}\mathcal{H}_0 &: x[n] = b + w[n] \quad n = 0, 1, 2, \ldots, N-1 \\ \mathcal{H}_1 &: x[n] = (-1)^n a + b + w[n] \quad n = 0, 1, 2, \ldots, N-1.\end{aligned}$$

Note that a and b are not known, and as before the noise is Gaussian distributed as $\mathbf{w} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$.

(2.5 pts) (c) Derive the generalized likelihood detector (GLRT) for this problem.

(2.5 pts) (d) Determine the false alarm probability P_{fa} and detection probability P_d. Give the P_{fa} and P_d as a function of the threshold γ and the error function of the test statistic.