Deptartment of Electrical Communication Engineering
Indian Institute of Science

E1 244 Detection and Estimation

6 March 2020, 13:45–15:15, Mid-term exam

This exam has two questions (20 points). Question 2 is on the back side of this page.

A4 cheat sheet is allowed. No other materials will be allowed.

Question 1 (10 points)

In wireless communications, fluctuation of the received signal amplitude is usually modeled using the Rayleigh distribution. In this question, we wish to estimate the amplitude by estimating the shape parameter of the Rayleigh distribution. The independent and identically distributed (iid) observations $x[n]$ for $n = 0, 1, \ldots, N-1$ have a Rayleigh PDF

$$p(x[n];\theta) = \begin{cases} \frac{x[n]}{\theta} \exp\left(-\frac{1}{2}\frac{x^2[n]}{\theta}\right) & x[n] > 0 \\ 0 & x[n] < 0 \end{cases}$$

and the Rayleigh CDF is given by

$$f(x;\theta) = 1 - \exp\left(\frac{-x^2}{2\theta}\right) \quad \text{for} \quad x \geq 0$$

where θ is the unknown parameter that we wish to estimate. Further, we have

$$E\{x[n]\} = \sqrt{\frac{\pi\theta}{2}} \quad \text{and} \quad \operatorname{var}\{x[n]\} = \left(2 - \frac{\pi}{2}\right)\theta.$$

(3pts) (a) Show that the regularity condition holds and find the Cramér-Rao lower bound for θ.

(2pts) (b) Find the sufficient statistic for θ and find the MVU.

(2pts) (c) Find the maximum likelihood estimator for θ.

(3pts) (d) Find the Cramér-Rao lower bound for $g(\theta) = \Pr\{x[n] > 1\}$, which is the probability that the nth observation $x[n]$ is greater than 1.

Question 2 (10 points)

Independent bivariate Gaussian samples $\{\mathbf{x}[0], \mathbf{x}[1], \ldots, \mathbf{x}[N-1]\}$ are observed. Each observation is a 2×1 vector that is distributed as $\mathbf{x}[n] \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ and

$$\mathbf{C}(\rho) = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

(2pts) (a) Find the Cramér-Rao lower bound (CRLB) for the correlation coefficient ρ.

Hint: You may want to use the following matrix identity:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and the Fisher information for the Gaussian case is given by

$$I(\theta) = \frac{1}{2} \mathrm{tr} \left\{ \left(\mathbf{C}^{-1}(\theta) \frac{\partial \mathbf{C}(\theta)}{\partial \theta} \right)^2 \right\}.$$

(1pt) (b) Plot the CRLB versus the correlation coefficient $\rho \in [-1, 1]$. Explain what happens when $\rho \to \pm 1$ and for $\rho = 0$.

(3pts) (c) Derive the least squares estimator (LSE) of ρ. The LSE is found by minimizing

$$J = \mathrm{tr} \left\{ [\mathbf{S} - \mathbf{C}(\rho)]^T [\mathbf{S} - \mathbf{C}(\rho)] \right\},$$

where

$$\mathbf{S} = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{x}[n] \mathbf{x}^T[n]$$

is the 2×2 sample covariance matrix.

Hint: The identities

$$\frac{\partial}{\partial x} \mathrm{tr}\{\mathbf{A}^T \mathbf{B}(x)\} = \mathrm{tr} \left\{ \mathbf{A}^T \frac{\partial \mathbf{B}(x)}{\partial x} \right\}$$

and

$$\frac{\partial}{\partial x} \mathrm{tr}\{\mathbf{B}^T(x) \mathbf{B}(x)\} = 2\mathrm{tr} \left\{ \mathbf{B}^T(x) \frac{\partial \mathbf{B}(x)}{\partial x} \right\}$$

will be useful.

(4pts) (d) Let $\mathbf{x}[0] \in \mathbb{R}^2$ denote the measurement of a DC level A in correlated Gaussian noise as

$$\mathbf{x}[0] = A\mathbf{1}_2 + \mathbf{w}[0]$$

where the covariance matrix of $\mathbf{w}[0]$ is $\mathbf{C}(\rho) \in \mathbb{R}^{2 \times 2}$. We will now assume ρ can be designed. Compute the Cramér-Rao lower bound for A and determine the value of ρ that minimizes the Cramér-Rao lower bound for A. Explain the effect that this ρ value has on the noise in the minimum variance unbiased estimator of $\hat{A}$.