Mathematical background II :

Roadmap :

① Sets : Open , closed , Compact

② Functions : Extend real value , Closedness, Continuity , Semi continuity , Lipschitz continuous

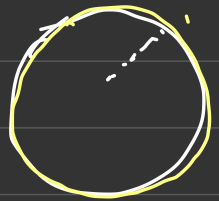
③ Optimization problem :
- Solution (minimum ; local , global)
- Existence of optimal solutions (Weierstrass' theorem)

Reference :

- Bertsekas , Non-linear programming (Appendix A)
- Beck , First -order optimization (ch. 2)

Open and closed sets:

Closed set: A set $A \subset \mathbb{R}^n$ that contains all of its limit points



Examples: $[0, 1]$, $B[c, \epsilon] = \{ \underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{c}\| \leq \epsilon \}$

$0 \leq x_i \leq 1$

Open set: Complement of closed set.

Examples: $(0, 1)$

$0 < x_i < 1$

$\underline{x} \in X \subseteq \mathbb{R}^n$

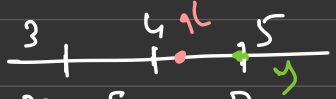
$\epsilon > 0$

$N_\epsilon(x) = \{ \underline{y} \in X : \|\underline{x} - \underline{y}\| < \epsilon \}$

$N_\epsilon(x) \subseteq X$

"All elements are interior points"

$(x - \epsilon, x + \epsilon) \subset X$



$X = [3, 5]$

Cannot take the end point

Closed set: $N_\epsilon(x) \subseteq X$ does not hold

Compact set :

- Bounded set : $\exists\ c \in \mathbb{R}$ such that the magnitude of any cordinate of any element of A is less than c

Example : $[2, \infty)$ not bounded $[2, 5]$ bounded

A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded

Example: Any closed interval $[a, b]$ in $\mathbb{R}$

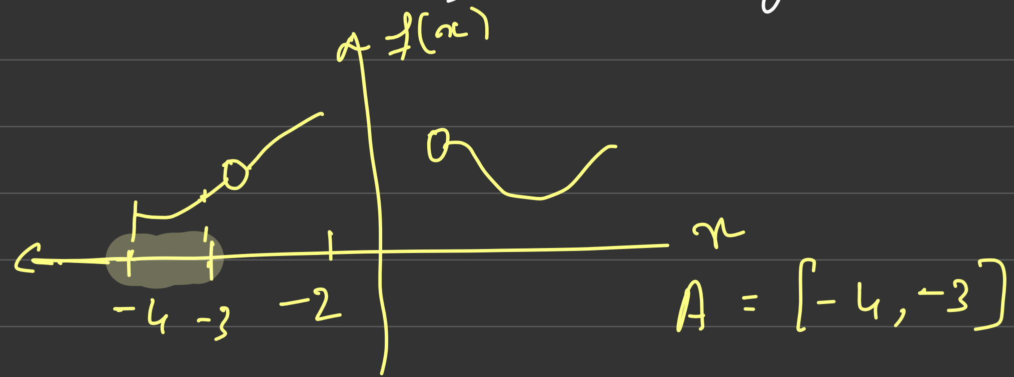
Continuity:

Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$

ⓐ f is continuous at a point $\underline{x} \in A$ if

$$\lim_{y \to x} f(y) = f(x)$$

continuous over A if it is continuous at every point $x \in A$



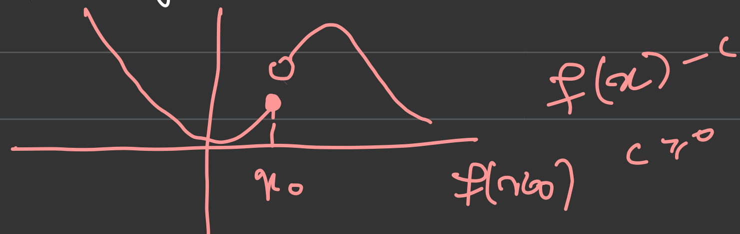
upper semicontinuity:

$$f(x) \geq \lim_{k \to \infty} \sup f(\underline{x}_k)$$

ⓑ $f : A \to \mathbb{R}$ is lower semicontinuous at $x \in A$ if

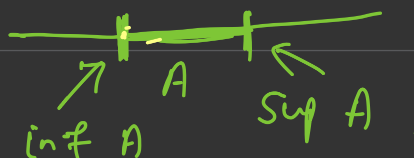
$$f(\underline{x}) \leq \lim_{k \to \infty} \inf f(\underline{x}_k)$$

for any sequence $\{\underline{x}_k\}$ of elements in A converging to $\underline{x}$



infimum:

largest scalar $x \leq y \;\; \forall y \in A$



(c) **Coercive:** $f: A \to \mathbb{R}$

$$\lim_{k \to \infty} f(x_k) = \infty$$

for every sequence $\{x_k\}$ of elements of A

such that $\|x_k\| \to \infty$

Extended real-valued functions:

defined over the entire underlying space $\mathbb{R}$ as well as $-\infty$ and ∞

$[-\infty, \infty]$

$$\boxed{\begin{aligned} 0 \cdot \infty &= \infty \cdot 0 = 0 \\ \alpha + \infty &= \infty \\ \alpha \cdot \infty &= \infty \cdot \alpha = \infty \end{aligned}}$$

Domain: set $f: \mathbb{R}^n \to \mathbb{R}$

$$\mathrm{dom}(f) = \{ \underline{x} \in \mathbb{R}^n : f(\underline{x}) < \infty \}$$

Example: Indicator function of $C \subseteq \mathbb{R}^n$

$$\delta_C(\underline{x}) = \begin{cases} 0 & \underline{x} \in C \\ \infty & x \notin C \end{cases}$$

with $\mathrm{dom}(\delta_C) = C$

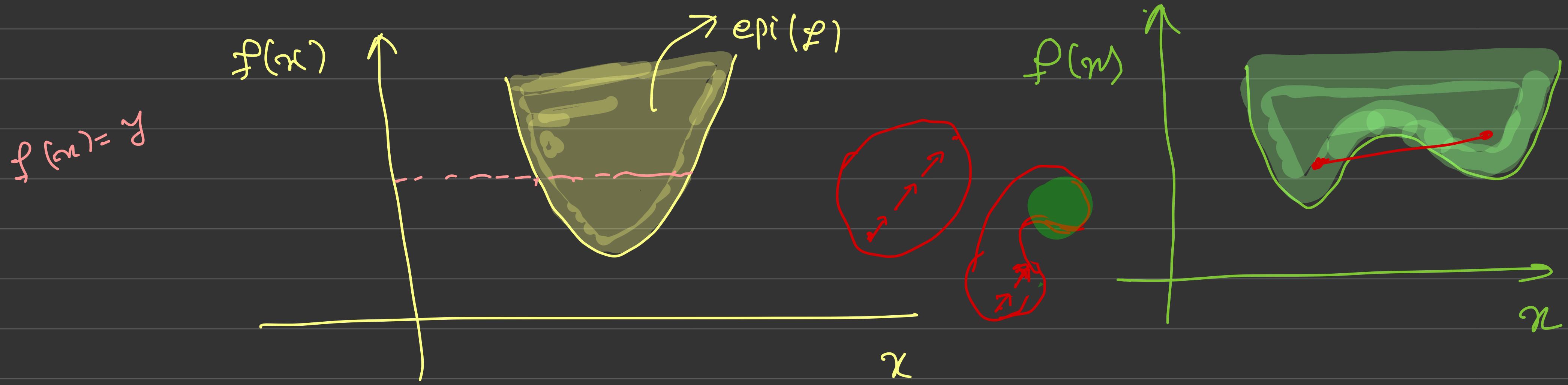
$f(x) = \log(ax + b)$

$\mathrm{dom}(f)$?

Epigraph:

$f : \mathbb{R}^n \to [-\infty, \infty]$

$$\text{epi}(f) = \{ (\underline{x}, y) : f(\underline{x}) \leq y ,\ \underline{x} \in \mathbb{R}^n, y \in \mathbb{R} \}$$
$$\subseteq \mathbb{R}^n \times \mathbb{R}$$



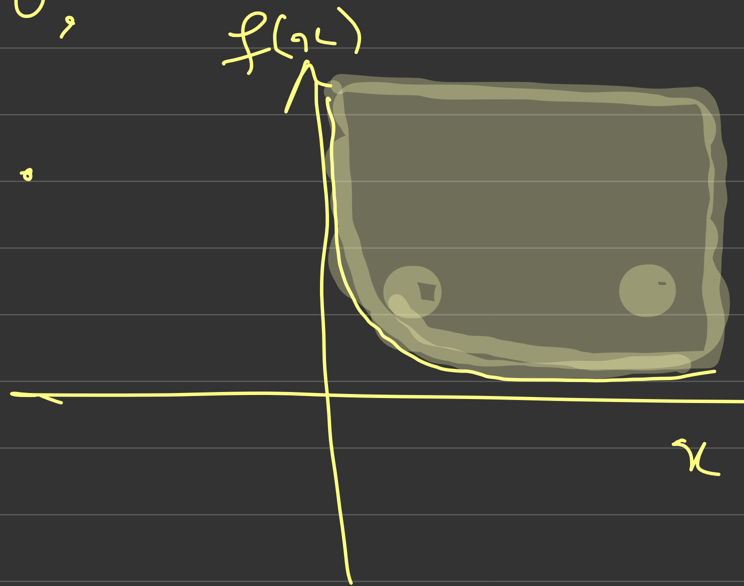
Closed functions: $f : \mathbb{R}^n \to [-\infty, \infty]$ is closed if its epigraph is closed

$$\text{epi}(\delta_C) = \{ (\underline{x}, y) \in \mathbb{R}^n \times \mathbb{R} : \delta_C(\underline{x}) \leq y \} = C \times \mathbb{R}_+$$

$\Longrightarrow$ $\text{epi}(\delta_C)$ is closed when C is closed

$\Longrightarrow$ if f is closed, $\mathrm{dom}(f)$ is not necessarily closed

$$f(x) = \begin{cases} \frac{1}{x}, & x > 0, \\ \infty, & \text{else}. \end{cases}$$



$\mathrm{dom}(f) = (0, \infty)$ open interval (not closed)

$\mathrm{epi}(f) = \{ (x, y) : xy \geq 1 ; x > 0 \}$

Closed set

So $f(x)$ is closed

$$\frac{1}{x} \leq y$$

$$1 \leq xy$$

For $f : \mathbb{R}^n \to [-\infty, \infty]$, the following are equivalent :

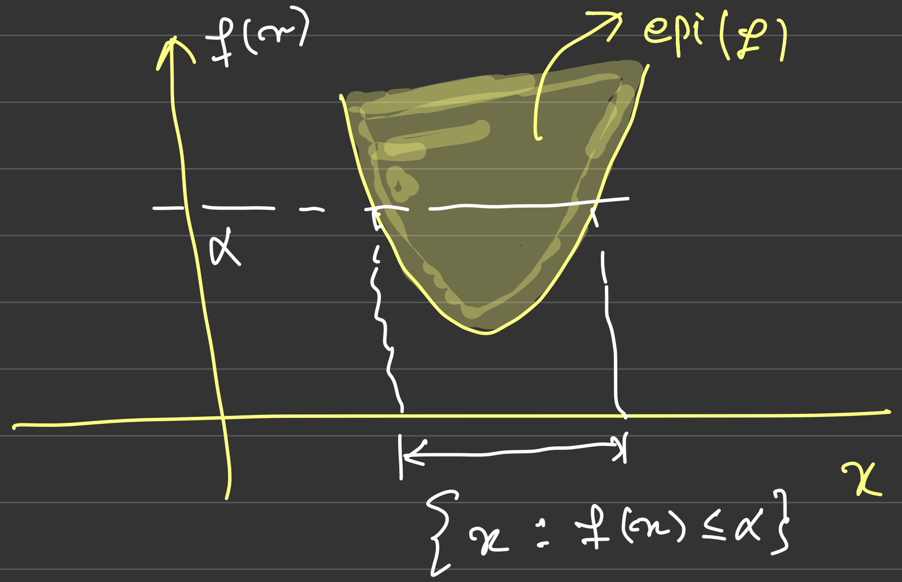
① f is lower semicontinuous

② f is closed

③ α-Sublevel set : For any $\alpha \in \mathbb{R}$

$$C_\alpha(f) = \{ \underline{x} \in \mathbb{R}^n : f(\underline{x}) \leq \alpha \}$$

is closed



Proof: lower s.c closed

- ① $\Rightarrow$ ② : Let $\{(\underline{x}_k, y_k)\} \subseteq \text{epi}(f)$ such that

$(\underline{x}_k, y_k) \to (x^*, y^*)$ as $k \to \infty$. Then

$$f(\underline{x}_k) \leq y_k$$ defn. epigraph.

By lower semi continuity,

$$f(\underline{x}^*) \leq \liminf_{k\to\infty} f(\underline{x}_k) \leq \liminf_{k\to\infty} y_k = y^*$$

So $(\underline{x}^*, y^*) \in \text{epi}(f)$. Hence f is closed ■

f closed C_α closed

- ② $\Rightarrow$ ③ Let $\{x_k\} \subset C_\alpha(f)$ and $x_k \to x^*$

$\boxed{\text{f is closed if epi(f) is closed}}$

$(\underline{x}_k, \alpha) \in \text{epi}(f)$ and $(\underline{x}_k, \alpha) \to (\underline{x}^*, \alpha)$

So $(\underline{x}^*, \alpha) \in \text{epi}(f)$ and $\underline{x}^* \in C_\alpha$

■

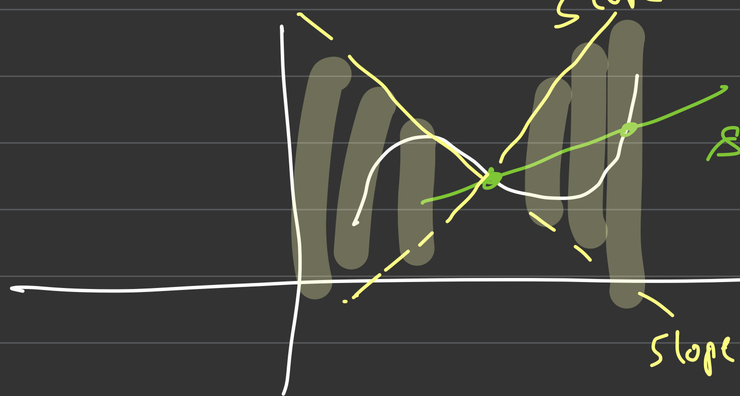
Lipschitz - continuous functions:

$f : X \longrightarrow \mathbb{R}^n$ is called Lipschitz continuous if there exists $B \geq 0$

$$\| f(\underline{x}) - f(\underline{y}) \| \leq B \, \| \underline{x} - \underline{y} \| \quad \forall \; \underline{x}, \underline{y} \in X$$

- B is the Lipschitz constant of f over X

$$\Rightarrow \quad \frac{\| f(\underline{x}) - f(\underline{y}) \|}{\| \underline{x} - \underline{y} \|} \leq B \quad ; \; \underline{x} \neq \underline{y}$$



slope $= \dfrac{f(x) - f(y)}{x - y}$

$f(x) - B\,|x-y| \leq f(y) \leq f(x) + B\,|x-y|$

Optimization problem:

$$\underset{\underline{x} \in \mathcal{X}}{\text{minimize}} \quad f(\underline{x})$$

$f : \text{dom}(f) \longrightarrow \mathbb{R}$ objective or cost function

$f(x)$: value of the objective function

$\mathcal{X} \subset \text{dom}(f)$ is the feasible set

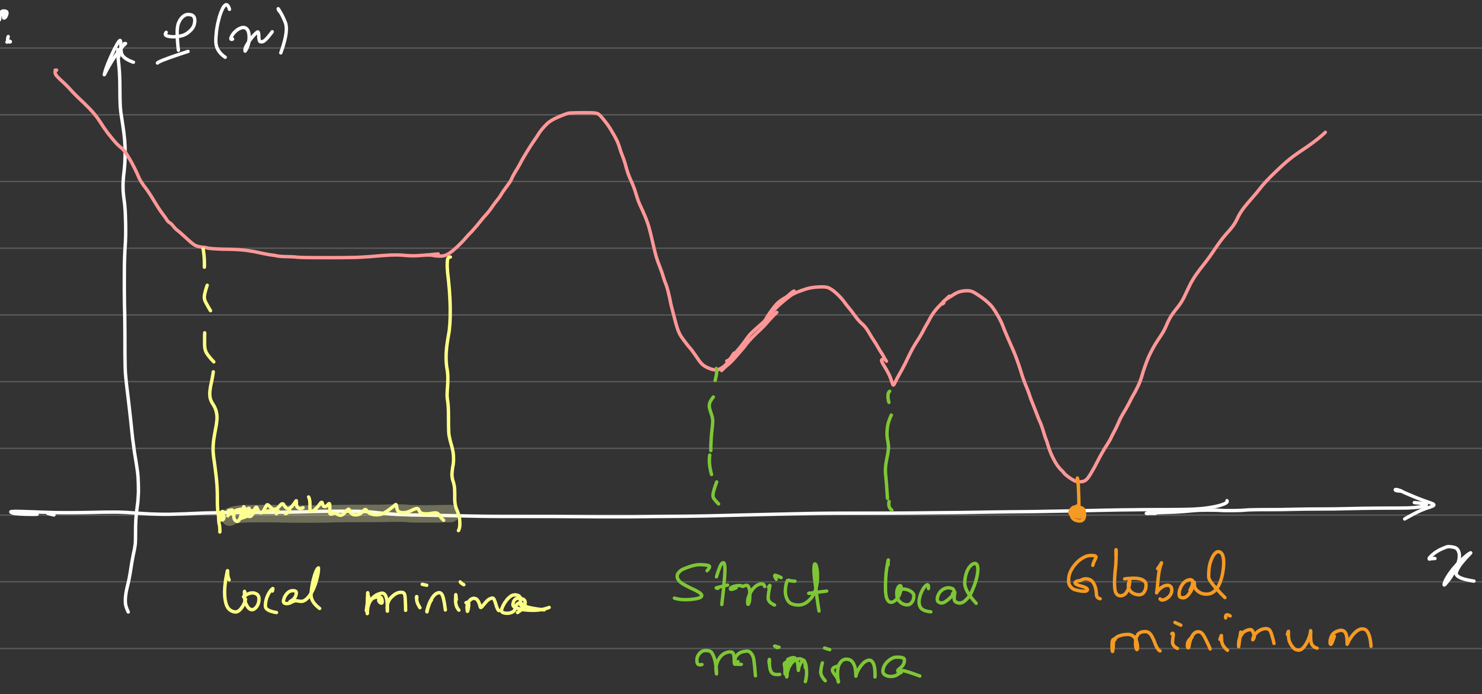
$\underline{x}$ is the variable

$$\text{minimize} \quad f(x_1, x_2)$$

$$\text{s.t.} \quad x_1 \geq 0, \; x_2 \geq 0$$

$$x_1 + x_2 \leq 1$$

Minima :



- $x^* \in X$ is a local minimum if $\exists\ \epsilon > 0$ such that $f(x) \geq f(x^*)\ \forall\ x \in X$ with $\|\underline{x} - \underline{x}^*\| < \epsilon$

- $x^* \in X$ is a strict local minimum if $\exists\ \epsilon > 0$ such that $f(x) > f(x^*)\ \forall\ x \in X$ with $\|\underline{x} - \underline{x}^*\| < \epsilon$

- $x^* \in X$ is a local minimum if $\exists\ \epsilon > 0$ such that $f(x) \geq f(x^*)\ \forall\ x \in X$

strict if $>$

$\underline{x} \neq \underline{x}^*$

Set of all global minima : $\underset{\underline{x} \in X}{\arg\min}\ f(x)$