Lecture 6   E1 260   Theory of convex functions

- Optimization problems :
  Convex , optimality criterion,
  Equivalent problems (rules)

  Canonical convex optimization problems:

  - Linear program , quadratic program,
    Semidefinite program,

    (linear fractional, QCQP, SOCP,
    GP)

Reference :
Boyd, Convex optimization [Chapter 4]

Optimization problems in Standard form:

$$\begin{aligned} \text{minimize} \quad & f_0(\underline{x}) \\ \text{Subject to} \quad & f_i(\underline{x}) \le 0 \quad i = 1, \dots, m \\ & h_i(\underline{x}) = 0 \quad i = 1, \dots, p \end{aligned}$$

$f_0 : \mathbb{R}^n \to \mathbb{R}$ objective function

$f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, m$ inequality constraint functions

$h_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, p$ equality constraint functions

Optimization domain : $D = \operatorname{dom}(f_0) \cap \bigcap_{i=1}^{m} \operatorname{dom}(f_i) \cap \bigcap_{i=1}^{p} \operatorname{dom}(h_i)$

Optimal value:

$$f^* = \inf \{ f_0(\underline{x}) : f_i(\underline{x}) \le 0, i = 1, \dots, m, \ h_i(\underline{x}) = 0, i = 1, \dots, p \}$$

$f^* = \infty$ [infeasible]

$f^* = -\infty$ if unbounded from below

$h_i(\underline{x}) = x_1^2 + x_2^2 = 5$ → not convex optimization

f_0 and f_i are convex, h_i are affine $[a_i^T \underline{x} = b_i]$ $\Rightarrow$ convex optimization problems!!

- if $\underline{x} \in D$; $f_i(x) \leq 0$ and $h_i(x) = 0$. Then $\underline{x}$ is called a <u>feasible point</u>
- if $\underline{x}$ is feasible and $f(x) \leq f^* + \epsilon$ then $\underline{x}$ is <u>ϵ - suboptimal</u>
- if $\underline{x}$ is feasible and $f_i(x) = 0$, then f_i is <u>active</u>

<u>unconstrained problem</u>:

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

has implicit constraints $\underline{a}_i^T \underline{x} < b_i$

Some important convex optimization problems in standard form:

Linear program (LP):

$$\underset{\underline{x}}{\text{minimize}} \quad \underline{c}^T \underline{x}$$

$$\text{s.t.} \quad A\underline{x} = b$$

$$\underline{x} \geq 0$$

Quadratic program:

$$\underset{\underline{x}}{\text{minimize}} \quad \frac{1}{2}\underline{x}^T Q \underline{x} + \underline{p}^T x + c$$

$f_i(x) \leq 0$

$$\text{s.t.} \quad A\underline{x} = b$$

$$\underline{x} \geq 0$$

Semidefinite program (SDP):

$$\underset{X}{\text{minimize}} \quad \mathrm{tr}(CX)$$

$$\text{s.t.} \quad \mathrm{tr}(A_i X) = b_i \quad i = 1, \ldots, p$$

$$X \succeq 0$$

- CVX
- SeDumi
- SDPT3

First-order Optimality conditions:

minimize $f_0(\underline{x})$
s.t. $f_i(\underline{x}) \le 0, \; i = 1, \ldots m$
$A\underline{x} = b$

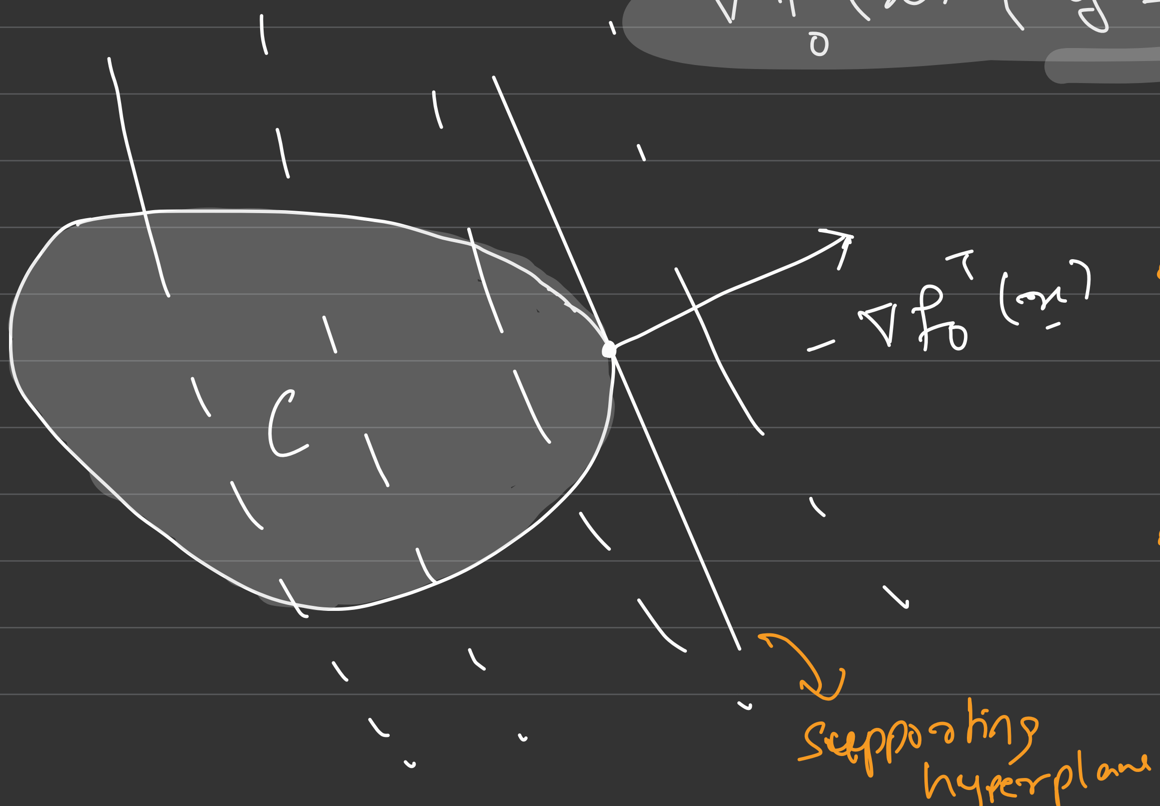
$\equiv$

minimize $f_0(\underline{x})$
s.t. $\underline{x} \in C$

$C = \{\underline{x} : f_i(\underline{x}) \le 0 ; i = 1, \ldots, m, \; A\underline{x} = b\}$

Suppose f_0 is differentiable. A feasible point $\underline{x}$ is optimal iff

$$\nabla f_0^T(\underline{x})(y - \underline{x}) \geq 0 \quad \forall \, y \in C$$



- All feasible directions are aligned with the gradient $\nabla f_0(\underline{x})$
- unconstrained $C = \mathbb{R}^n$

$\Rightarrow \nabla f(\underline{x}) = 0$

Proof:

Suppose $\underline{x} \in C$ and $\nabla f_0^T(x)(y-x) \geq 0$

Then $\underline{y} \in C$

$$f_0(\underline{y}) \geq f_0(x) + \underbrace{\nabla f_0^T(x)(\underline{y} - \underline{x})}_{\geq 0}$$

$\Rightarrow$ $f_0(y) \geq f_0(x)$ $\Rightarrow$ $\underline{x}$ is an optimal point

Conversely:

Suppose x is optimal

and $\nabla f_0^T(x)(y-x) < 0$.

Consider $g(t) = ty + (1-t)x \quad t \in [0,1]$

$\Rightarrow$ $g(t)$ is feasible

$$\frac{d}{dt} f_0(g(t))\Big|_{t=0} = \nabla f_0^T(x)(y-x) < 0$$

So for small positive t, we have $f_0(g(t)) < f_0(x)$

Unconstrained problem:

$$m = 0 \quad , \quad p = 0$$

$$\nabla f_0^T(x)(\underline{y} - \underline{x}) \geq 0$$

f_0 is differentiable (dom(f_0) is open)

Suppose we take $\underline{y}$ close to $\underline{x}$

$$\underline{y} = \underline{x} - t \nabla f_0(x) \qquad \text{for small 't'}$$

$$\nabla f_0^T(\underline{x})(\underline{y} - \underline{x})$$

$$= -t \, \| \nabla f_0(\underline{x}) \|_2^2 \geq 0$$

$$\Rightarrow \qquad \nabla f_0(x) = 0$$

Example:

Equality Constrained problem:

minimize $f_0(x)$

s.t. $A\underline{x} = \underline{b}$

$\nabla f_0^T(\underline{x})(\underline{y} - \underline{x}) \geq 0 \quad \forall\, y : A\underline{y} = b$

Since x is feasible, every feasible

$$\underline{y} = x + v \quad , \quad v \in N(A)$$
$$A\underline{v} = 0$$

$$\Rightarrow \nabla f_0^T(x)\,\underline{v} \geq 0 \quad \forall\, \underline{v} \in N(A)$$

If a linear fn. is nonnegative on a subspace, then it must be zero on the subspace.

$$\Rightarrow \nabla f_0^T(\underline{x})\,\underline{v} = 0 \quad \forall\, \underline{v} \in N(A) \Rightarrow \nabla f_0(x) \perp N(A)$$

Since $N(A)^\perp = R(A^T)$

$$\Rightarrow \nabla f_0(x) + A^T \underline{v} = 0$$

Lagrange multiplier optimality condition:

$$\min_{\underline{x},\, \underline{v}} \quad f(\underline{x}) + v^T (A\underline{x} - b)$$

$$\nabla f(x) + A^T \underline{v} = 0$$

Equivalent problems:

① Transformation and change of variables:

$$\underset{x \in C}{\text{minimize}} \quad f(x) \iff \underset{x \in C}{\text{minimize}} \quad h(f(x))$$

- $h : \mathbb{R} \to \mathbb{R}$ is a monotone increasing function

$$\underset{x \in C}{\text{minimize}} \quad f(x) \iff \underset{\phi(y) \in C}{\text{minimize}} \quad f(\phi(y))$$

- $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one with its image covering C

Example: 1 $\underset{\mu}{\text{minimize}} \quad -e^{-\sum_n (x_n-\mu)^2}$

$g(z) := \log(-z)$

$\equiv \underset{\mu}{\text{minimize}} \quad \sum_n (x_n-\mu)^2$

2 $\underset{x_1, x_2 \in \mathbb{R}}{\text{minimize}} \quad f(x_1, x_2)$

s.t. $\frac{x_1}{x_2} - 4 \leq 0$: not convex

$x_2 \geq 2$

writing $\frac{x_1}{x_2} - 4 \leq 0$ $\quad x_1 - 4x_2 \leq 0$

Eliminating equality constraints:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0 \quad i = 1, \dots, m \\ & Ax = b \end{aligned}$$

$$\left.\begin{aligned} & Ax = b \\ & Ax_0 = b \\ & x = x_0 + Fz \\ & \text{null}(A) = \text{range}(F) \end{aligned}\right\}$$

$$\iff \quad \begin{aligned} \underset{z}{\text{minimize}} \quad & f_0(Fz + x_0) \\ \text{s.t.} \quad & f_i(Fz + x_0) \le 0 \quad i = 1, \dots, m \end{aligned}$$

Epigraph form:

$$\begin{aligned} \underset{\underline{x}}{\text{minimize}} \quad & f_0(\underline{x}) \\ \text{s.t.} \quad & f_i(\underline{x}) \le 0 \quad i=1,\dots,m \\ & A\underline{x} = b \end{aligned} \iff \begin{aligned} \underset{x,\, t}{\text{minimize}} \quad & t \\ & f_0(x) - t \le 0 \\ & f_i(x) \le 0 \\ & A\underline{x} = b \end{aligned}$$

Example:

$$\underset{\underline{x} \in C}{\text{minimize}} \quad \underline{v}^T A^{-1}(x)\, \underline{v} \qquad A(x) \succ 0$$

$$\iff \underset{\underline{x} \in C,\, t}{\text{minimize}} \quad t$$

$$t \geq \underline{v}^T A^{-1}(x)\, v$$

$$t \geq \underline{v}^T A^{-1}(x)\, \underline{v} \iff \begin{bmatrix} t & \underline{v}^T \\ \underline{v} & A(x) \end{bmatrix} \succeq 0$$