Department of Electrical Communication Engineering
Indian Institute of Science

E9 211 Adaptive Signal Processing

3 Dec 2020, Take-home mid-term assessment

This exam has two questions (20 points).

This is an open book assessment with a turn in time of 24 hrs. Make sure the report is turned in before 9 am, 4 December 2020. This is a hard deadline. Your report should be a single PDF file, legible, scanned/pictured under good lighting conditions with your full name. The report should be submitted via MS Teams. Late or plagiarized submissions will not be graded.

Question 1 (10 points)

Suppose we receive signals from two interfering sources s_1 and s_2 in noise as

$$\mathbf{x} = \mathbf{a}_1 s_1 + \mathbf{a}_2 s_2 + \mathbf{n},$$

where the column vector $\mathbf{x}$ collects observations from an array of length M. The array steering (column) vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are assumed to be known. The receiver noise $\mathbf{n}$ is assumed to be zero mean with covariance matrix $\sigma^2\mathbf{I}$. Here, $\mathbf{I}$ is the identity matrix of size M. We assume that s_1, s_2, and noise are mutually uncorrelated. The sources have unit power.

We are interested in recovering the source symbols s_1 using a linear receiver as

$$\hat{s}_1 = \mathbf{w}^H \mathbf{x}$$

using the following constrained beamformers.

(4pts) (a) Find a receiver $\mathbf{w}$ that has a unity gain (distortionless response) towards the direction of source s_1 and minimizes the mean squared error. Show that this receiver minimizes the interference plus noise power at the output of the beamformer, where we treat s_2 as interference.

(4pts) (b) Find a unit norm beamformer, i.e., $\|\mathbf{w}\|_2 = 1$, that maximizes the signal to interference plus noise ratio, where the signal and interference plus noise components in $\mathbf{w}^H\mathbf{x}$ are $\mathbf{w}^H\mathbf{a}_1\mathbf{a}_1^H\mathbf{w}$ and $\mathbf{w}^H[\mathbf{a}_2\mathbf{a}_2^H + \sigma^2\mathbf{I}]\mathbf{w}$, respectively.

(2pts) (c) Compare the beamformers computed in part (a) and (b) of this question in terms of the signal to interference plus noise ratio.

Question 2 (10 points)

Consider a first-order autoregressive, i.e., AR(1), process $x(n)$ that has an autocorrelation sequence

$$r_x(k) = \alpha^{|k|}.$$

We make noisy measurements of $x(n)$ as

$$y(n) = x(n) + v(n)$$

where $v(n)$ is zero mean white noise with a variance of σ^2 and $v(n)$ is uncorrelated with $x(n)$. We find the optimum first-order linear predictor of the form

$$\hat{x}(n+1) = w(0)y(n) + w(1)y(n-1) = \mathbf{w}^H\mathbf{y}$$

where $\mathbf{w} = [w(0)\ w(1)]^T$ and $\mathbf{y} = [y(n)\ y(n-1)]^T$.

(3pts) (a) Derive the optimum $\mathbf{w}$ by minimizing the mean-squared error

$$J = E\{|\hat{x}(n+1) - x(n+1)|^2\}.$$

What happens to $\mathbf{w}$ in the noise-free case $\sigma^2 \to 0$. Why?

(3pts) (b) Develop a steepest gradient descent algorithm that determines $\mathbf{w}$ iteratively. Provide a condition on the step-size μ in terms of α in order to guarantee convergence. Provide the value of the step-size that yields fastest convergence, and also the resulting time-constant.

(4pts) (c) Consider a modified cost function

$$J = E\{|\hat{x}(n+1) - x(n+1)|^2\} + \beta\|\mathbf{w}\|^2.$$

Design a steepest gradient descent algorithm that determines $\mathbf{w}$ iteratively. Find the value of the step-size that yields fastest convergence and compare with the optimum μ from the previous question. Comment on the convergence rate (i.e., the time constant) for $\beta > 0$ and argue when is the modified cost function useful.