Department of Electrical Communication Engineering
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E9 211 Adaptive Signal Processing

28 January 2021, Take-home final assessment

This exam has two questions (20 points).

This is an open book assessment with a turn in time of 24 hrs. Make sure the report is turned in before 9 am, 29 January 2021. This is a hard deadline. Your report should be a single PDF file, legible, scanned/pictured under good lighting conditions with your full name. Use `firstname_lastname.pdf` as the file name. The report should be submitted via MS Teams. Late or plagiarized submissions will not be graded.

Question 1 (10 points)

In decision-directed feedback equalization using Viterbi decoder, the desired signal and thus the error is not available until a number of samples later. Suppose the signal $\mathbf{x}(k)$ is real-valued, we consider an LMS algorithm with the update equation

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \mu e(k-1)\mathbf{x}(k-1)$$

where the error signal is given by

$$e(k-1) = d(k-1) - y(k-1) = d(k-1) - \mathbf{w}_{k-1}^T \mathbf{x}(k-1).$$

Here, $d(k)$ is the desired signal. This LMS algorithm works with signals delayed by one sample. Without any delay, this would be the conventional LMS algorithm.

(5pts) (a) Find the values of μ for which this LMS algorithm converges in the mean.

Hint: The solution of a second-order difference equation converges to zero when the roots associated with its characteristic equation lie within the unit circle.

(5pts) (b) Determine the slowest decaying mode when the eigenvalues of $\mathbf{R}_x = E\{\mathbf{x}(k)\mathbf{x}(k)\}$ are all one and we choose $\mu = 0.5$. Find the time constant for the LMS algorithm.

Solutions

(a) Taking the expected value of the update equation, we have

$$E\{\mathbf{w}_{k+1}\} = E\{\mathbf{w}_k\} - \mu \mathbf{R}_x E\{\mathbf{w}_{k-1}\} + \mu \mathbf{r}_{xd}$$

where we assume that the weight vector $\mathbf{w}_k$ is uncorrelated with the data vector $\mathbf{x}(k)$. Therefore the expected value of the weight vector (thus the error vector) satisfies a second-order difference equation. On diagonalizing the autocorrelation matrix, the coefficients of the transformed vector can be written as

$$E\{v_{k+1}(n)\} = E\{v_k(n)\} - \mu\lambda_n E\{v_{k-1}(n)\} + \mu r_{xd}(n)$$

where λ_n is the nth eigenvalue of $\mathbf{R}_x$.

The characteristic equation for $E\{v_k(n)\}$ is given by

$$1 - z^{-1} + \mu\lambda_n z^{-2} = 0.$$

For $E\{v_k(n)\}$ to converge in the mean, the roots of the characteristic equation must lie inside the unit circle. Since the roots are at

$$z_n = 0.5 \pm 0.5\sqrt{1 - 4\mu\lambda_n},$$

the delayed LMS algorithm converges in the mean provided

$$0 < \mu < \lambda_{\max}^{-1}.$$

(b) With $\lambda_n = 1$ and $\mu = 0.5$, the roots of the characteristic equation are

$$z_n = 0.5 \pm 0.5j.$$

Thus the envelope of all the modes behave as $0.5^{k/2}$ where k is the iteration index. The time constant τ is obtained as

$$0.5^{\tau/2} = 1/e \Rightarrow \tau = 2.885 \quad \text{(time steps)}.$$

Question 2 (10 points)

An autoregressive process of order 1 is described by the difference equation

$$x(n) = 0.5x(n-1) + w(n)$$

where $w(n)$ is zero-mean white noise with a variance $\sigma_w^2 = 0.64$. The observed process $y(n)$ is described by

$$y(n) = x(n) + v(n)$$

where $v(n)$ is zero-mean white noise with a variance of $\sigma_v^2 = 1$.

(2pts) (a) Write the Kalman filtering equations to find the minimum mean-square estimate, $\hat{x}(n|n)$, of $x(n)$ given the observations $y(i), i = 1, \ldots, n$. The initial conditions are $\hat{x}(0|0) = 0$ and $E\{|x(0) - \hat{x}(0|0)|^2\} = 1$.

(3pts) (b) Assuming that the filter reaches a steady state solution, find the steady state covariance and the steady state Kalman gain. In the steady state, $P(n+1|n+1) = P(n|n)$. Find the steady state estimation equation for $\hat{x}(n|n)$.

(5pts) (c) Now consider the autoregressive moving average ARMA(1,1) process

$$y(n) + ay(n-1) = w(n) + bw(n-1).$$

Give the state-space model for the ARMA process and write the Kalman filtering equations.

Solutions

(a) For the state-space model

$$\begin{aligned} x(n) &= 0.5x(n-1) + w(n) \\ y(n) &= x(n) + v(n) \end{aligned}$$

with $A = 0.5$ and $C = 1$, the Kalman filtering algorithm initialized with $\hat{x}(0) = 0$, $P(0|0) = 1$, is

(a) $\hat{x}(n|n-1) = 0.5\hat{x}(n-1|n-1)$

(b) $P(n|n-1) = 0.25P(n-1|n-1) + 0.64$

(c) $K(n) = P(n|n-1)[P(n|n-1) + 1]^{-1}$

(d) $P(n|n) = [1 - K(n)]P(n|n-1)$

(e) $\hat{x}(n|n) = \hat{x}(n|n-1) + K(n)[y(n) - \hat{x}(n|n-1)]$

(b) We have

$$\begin{aligned} P(n|n) = [1 - K(n)]P(n|n-1) &= [1 - P(n|n-1)[P(n|n-1) + 1]^{-1}]P(n|n-1) \\ &= \frac{P(n|n-1)}{1 + P(n|n-1)}. \end{aligned}$$

Thus

$$P(n|n) = \frac{0.25P(n-1|n-1) + 0.64}{0.25P(n-1|n-1) + 1.64}$$

Since in the steady state, we have $P(n+1|n+1) = P(n|n)$, we have

$$P(n|n) = \frac{0.25P(n|n) + 0.64}{0.25P(n|n) + 1.64}.$$

This simplifies to $0.25P^2(n|n) + 1.39P(n|n) - 0.64 = 0$, which has a positive root $P(n|n) = 0.4276$. Therefore,

$$\hat{x}(n) = 0.5\hat{x}(n-1) + 0.4276[y(n) - 0.5\hat{x}(n-1)].$$

(c) Let

$$\mathbf{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}.$$

Then

$$\begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = \begin{bmatrix} -a & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(n-1) \\ x_2(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ b \end{bmatrix} w(n)$$

and

$$y(n) = [1 \quad 0]\mathbf{x}(n)$$

is the state-space representation for the ARMA(1,1) process. Here $\mathbf{Q}_v = \mathbf{0}$ and

$$\mathbf{Q}_w = \sigma_w^2 \begin{bmatrix} 1 & b \\ b & b^2 \end{bmatrix}.$$

Given the above state-space model, the Kalman filtering equations are standard.