Indian Institute of Science
Department of Electrical Communications Engineering

E9 211: Adaptive Signal Processing

October-January 2020

Homework 1 (deadline 27 Nov. 5 pm)

This homework consists of two parts: (a) generating and studying the source separation using an antenna array and (b) deriving a convolutive model with a single source and a single receiver. Make a short report containing the required Matlab/Python files, plots, explanations, and answers, and turn it in by the deadline using Microsoft Teams.

Part A: Antenna beamforming

Reference: Section 6.5, Adaptive Filters, Ali. H. Sayed; (or, Section 3.4, Fundamentals of Adaptive Signal Processing, Ali. H. Sayed).

1. Make Matlab subroutines to

   (a) generate the array response

   $$\mathbf{a}(\theta) = \begin{bmatrix} 1 \\ e^{j2\pi\Delta\sin\theta} \\ e^{j4\pi\Delta\sin\theta} \\ \vdots \\ e^{j2\pi\Delta(M-1)\sin\theta} \end{bmatrix} : M \times 1$$

   of a uniform linear array with M elements and spacing Δ wavelengths to a source coming from direction θ degrees;

   ```
   function a = gen_a(M,Delta,theta)
   ```

   (b) plot the spatial response $|y| = |\mathbf{w}^{\mathrm{H}}\mathbf{a}(\boldsymbol{\theta})|$ of a given beamformer $\mathbf{w}$ as a function of the direction θ of a source with array response $\mathbf{a}(\theta)$;

   ```
   function y = spat_response(w,Delta,theta_range)
   ```

   (c) generate a data matrix $\mathbf{X} = \mathbf{A}_\theta\mathbf{S}+\mathbf{N}$ as function of the directions $\boldsymbol{\theta} = [\theta_1 \ \cdots \ \theta_d]^{\mathrm{T}}$, number of antennas M, number of samples N, and signal-to-noise ratio (SNR) in dB (the SNR is defined as the ratio of the source power of a single user over the noise power). Here,

   $$\mathbf{A}_\theta = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_d)] : M \times d.$$

   The source symbols $\mathbf{S} : d \times N$ are chosen uniformly at random from a QPSK alphabet $\{(\pm 1 \pm j)/\sqrt{2}\}$. The noise matrix $\mathbf{N} : M \times N$ is random zero-mean complex Gaussian matrix. To generate a complex noise with variance σ^2, you should use the Matlab command: `(randn(M,N) + 1j*randn(M,N))*(σ/sqrt(2))`.

```
function X = gen_data(M,N,Delta,theta,SNR)
```

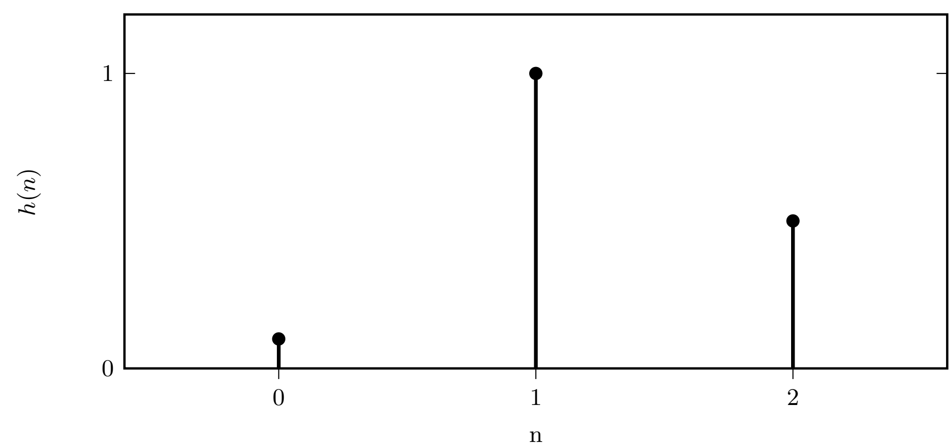
2. Test your routines and plot the spatial response for $\mathbf{w} = [1, 1, \cdots 1]^{\mathrm{T}}$ with $\Delta = 0.5$ and $M = \{2, 4, 7\}$ and with $M = 7$ and $\Delta = \{0.5, 1, 2\}$. Comment your observations.

3. Plot the singular values of $\mathbf{X}$ using Matlab command `plot(1:M, singular_values, '*')`. Investigate the behavior of the singular values for varying direction of arrival (DOA) separation of two sources, e.g., "separation" of 5 and 60 degrees, number of antennas $M \in \{7, 20\}$, $\Delta = 0.5$, number of samples $N \in \{20, 100\}$, SNR 0 and 20 dB. Now repeat the same exercise based on the eigenvalues of $\hat{\mathbf{R}}_x = \frac{1}{N}\mathbf{X}\mathbf{X}^{\mathrm{H}}$.

4. Consider a system with two sources and take $\boldsymbol{\theta} = [0°, 5°]^{\mathrm{T}}$, $M = 5$, $\Delta = 0.5$, and $N = 1000$. Now compute the matched filter $\mathbf{W}^{\mathrm{H}} = \mathbf{A}$, the zero-forcing receiver $\mathbf{W}^{\mathrm{H}} = \mathbf{A}^{\dagger}$ and the Wiener (i.e., LMMSE) receiver for the first source (we do not view the second source as noise), assuming the mixing matrix $\mathbf{A}_{\theta}$ and the noise variance σ^2 are known. For each of these beamformers, plot the estimated symbols in the complex plane for a few SNR values, such that you observe four clusters (use `plot(s_est,'x')`). From these plots, what can you conclude about the performance of these three beamformers?

Part B: Channel estimation and equalization

Reference: Sections 5.2, 5.4, and 6.3 Adaptive Filters, Ali. H. Sayed; (or, Sections 2.7.1, 2.7.3 and 3.2, Fundamentals of Adaptive Signal Processing, Ali. H. Sayed).



Finite impulse response (FIR) channel $H(z) = 0.1 + z^{-1} + 0.5z^{-2}$ with $L = 3$ taps.

1. Make Matlab subroutines to

   (a) construct a source sequence $\mathbf{s} = [s_0\ s_1\ \cdots\ s_{N-1}]^{\mathrm{T}}$, where every entry is a random QPSK symbol:

   ```
   function s = source(N)
   ```

   (b) construct the noisy received sequence $\mathbf{x} = [x_0\ x_1\ \cdots\ x_{N-1}]^{\mathrm{T}}$ obtained at the output of the FIR channel (see the channel impulse response above) with $L = 3$

taps $\mathbf{h} = [0.1, 1, 0.5]^{\mathrm{T}}$ as $\mathbf{x} = \mathbf{h} * \mathbf{s} + \mathbf{n}$. The receiver noise vector $\mathbf{n} : N \times 1$ is zero-mean complex Gaussian with variance σ^2;

```
function x = gen_data1(h,s,SNR)
```

Hint: You can use the Matlab `filter` command. If you use the Matlab command `conv` you will need to truncate the tail. `SNR` in dB will be $10 \log_{10} \frac{\|\mathbf{h}\|_2^2}{\sigma^2}$

2. Assume that $\mathbf{s}$ is known. How can we estimate the channel $\mathbf{h}$ from $\mathbf{x}$ and $\mathbf{s}$ using a minimum variance unbiased estimator? To do so, first derive how $\mathbf{x}$ can be decomposed as $\mathbf{x} = \mathbf{S}_{\text{pilot}}\mathbf{h} + \mathbf{n}$. Then construct the matrix $\mathbf{S}_{\text{pilot}}$ from the data sequence $\mathbf{s}$ and compute an estimate of $\mathbf{h}$ from $\mathbf{x}$ and $\mathbf{S}_{\text{pilot}}$. Plot the channel estimates and compare it to the true channel for $N = 1000$ and $\text{SNR} = \{10, 100\}$ dB. What can you conclude?

3. To estimate the source sequence, you need to construct a linear equalizer with M taps. To do so, first construct a data matrix

$$\mathbf{X} = [\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{N-1}] : M \times N$$

with $\mathbf{x}_i = [x_i, x_{i-1}, \cdots, x_{i-M-1}]^{\mathrm{T}}$. The data matrix can be written as $\mathbf{X} = \mathbf{HS} + \mathbf{N}$ where the

$$\mathbf{S} = [\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_{N-1}] : (M + L - 1) \times N$$

with $\mathbf{s}_i = [s_i, s_{i-1}, \cdots, s_{i-(M+L)-1}]^{\mathrm{T}}$ is constructed using the data sequence $\mathbf{s}$. Use $s_i = 0$ for $i < 0$. Derive the structure of the matrix $\mathbf{H}$ and construct $\mathbf{H} : M \times (M + L - 1)$ using the known channel taps $\mathbf{h}$. Compute the Wiener receiver (or LMMSE equalizer) for each row of $\mathbf{S}$, assuming $\mathbf{H}$ and the noise variance σ^2 are perfectly known. As before, plot the estimated symbols of each row (separately) in the complex plane for SNR values of 10 dB and 100 dB, $M = 5$ and $N = 1000$. Which row can we detect the best and why?