Bounds on the Sum-Rate Capacity of the Gaussian MIMO X Channel

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***Abstract*— We consider the MIMO X channel (XC), a system consisting of two transmit-receive pairs, where each transmitter communicates with both the receivers. Both the transmitters and receivers are equipped with multiple antennas. First, we derive an upper bound on the sum-rate capacity of the MIMO XC under individual power constraint at each transmitter. The sum-rate capacity of the two-user multiple access channel (MAC) that results when receiver cooperation is assumed forms an upper bound on the sum-rate capacity of the MIMO XC. We tighten this bound by considering noise correlation between the receivers and deriving the worst noise covariance matrix. It is shown that the worst noise covariance matrix is a saddle-point of a zero-sum, two-player convex-concave game, which is solved through a primal-dual interior point method that solves the maximization and the minimization parts of the problem simultaneously. Next, we propose an achievable scheme which employs dirty paper coding at the transmitters and successive decoding at the receivers. We show that the derived upper bound is close to the achievable region of the proposed scheme at low to medium SNRs.**

keywords: *MIMO X channel, interference channel, convex-concave game, primal-dual interior point method, dirty paper coding.*

I. Introduction

The capacity of wireless multiple-input multiple-output (MIMO) channels has attracted a lot of interest. The capacity region of point-to-point MIMO channel, MIMO multiple access channel (MAC), and that of the MIMO broadcast channel (BC) are characterized in [1]–[7]. In the two-user interference channel (IC), there are two transmitter–receiver pairs, where each transmitter intends to communicate with its corresponding receiver. Recently, in [8], the capacity of the Gaussian interference channel is characterized to within one bit.

The MIMO X channel (XC) is a generalization of the interference channel; there are two transmitter–receiver pairs, and each transmitter intends to communicate with both the receivers. Both the transmitters are equipped with multiple antennas. It is interesting to note that the MAC, BC and IC are contained within the MIMO XC and can be obtained as special cases of the XC.

Recently, the X channel has attracted considerable research interest [10,12]–[19]. The degrees of freedom of the MIMO X channel is found in [10], and it is shown to be $\frac{4M}{3}$, with $M > 1$ antennas at each node. It is shown that the concept of interference alignment (IA) coupled with zero forcing achieves the highest number of degrees of freedom. For the case of $M = 1$ antennas at the transmitters/receivers, the degrees of freedom is shown to be bounded above by $4/3$ and bounded below by 1. It was later shown in [11] that $4/3$ is indeed the degrees of freedom for the $M = 1$ case and introduced the novel idea of asymmetric complex signaling to achieve the outer bound. In [12], the authors combine dirty paper coding (DPC), zero forcing (ZF) and successive decoding methods to obtain signaling schemes which achieve the highest multiplexing gain or the degrees of freedom. They eventually transform the XC into four parallel channels. We refer to this scheme as the (MMK) (Maddah-Ali-Motahari-Khandani) scheme. A gradient projection based IA for the MIMO XC is developed in [14]. Algebraic expressions are derived to obtain a locally optimum IA solution with the objective of maximizing a utility of transmit rates. In [15], MMSE precoding algorithm to maximize the weighted sum-rate of the MIMO XC is designed through alternating optimization. Fixed-rate transmission schemes are developed in [16] using a combination of Alamouti codes and IA. In [17], linear IA transmit filters and ZF receive filters are designed for the XC, based on generalized singular value decomposition. In [18], the authors propose a perfect IA scheme for the K-user MIMO X network, a system consisting of K transmitters and K receivers, where all transmitters send independent messages to all receivers. Finally, space-time precoders with full diversity and low decoding complexity for XC is investigated in [19].

In this work, in the first part of the paper, we derive an upper bound on the sum-rate capacity of the Gaussian MIMO XC. Consider a Gaussian MIMO XC with separate power constraint at each transmitter. By assuming cooperation among the receivers, we get a Gaussian MIMO MAC channel with individual power constraint at each transmitter whose capacity region is characterized in [6,22]. Since the MIMO MAC channel is a MIMO XC with receiver cooperation, the capacity of the MIMO MAC channel is an upper bound on the capacity of the MIMO XC. This upper bound can be further tightened by considering noise correlation among the two receivers. This amounts to finding the worst noise covariance matrix for the MAC which gives a much stronger bound. However, finding the least favorable noise covariance matrix is a non-trivial problem as it involves both a maximization over the input covariance matrices and a minimization over the noise covariance matrices. It is shown that the worst noise covariance matrix is a saddle-point of a zero-sum, two-player convex-concave game, which is solved through a primal-dual interior point method that solves the maximization and the minimization parts of the problem

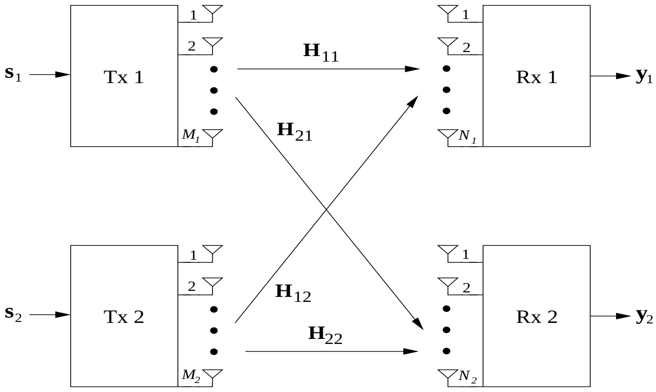

Fig. 1. MIMO X channel system model.

simultaneously. We argue that although the form of the MAC upper bound with worst noise correlation problem is similar to the sum capacity BC problem, unlike the latter case, the former problem presents significant difficulties in converting it to a single convex minimization problem, thus justifying the use of primal-dual interior point method to directly solve the former minimax problem.

In the second part of the paper, we propose an MMK like scheme which uses a combination of DPC and successive decoding and optimize over the encoding and decoding orders at the transmitters and receivers with the objective of maximizing the sum-rate capacity of the MIMO XC. We term this the *modified MMK (m-MMK) scheme*. Finally, we compare the upper bound with the achievable region of the proposed m-MMK scheme, and show that the upper bound is close to the achievable region of the m-MMK scheme for low to medium SNRs.

The rest of this paper is organized as follows. The system model is presented in Section II. The MAC upper bound problem is formulated and solved in Section III. The m-MMK scheme is presented in Section IV. Simulation results are discussed in Section V. Conclusions are presented in Section VI.

II. System Model

In this section, we describe the model to be used in the rest of the paper. We consider a MIMO system with two transmitters and two receivers as shown in Fig. 1. Transmitter t is equipped with M_t antennas, $t = 1, 2$, and receiver r, is equipped with N_r antennas, $r = 1, 2$. We assume a flat-fading environment. Let $\mathbf{H}_{rt} = [h_{ij}]$ denote[1] the $N_r \times M_t$ channel gain matrix from transmitter t to receiver r, where h_{ij} is the channel gain from the jth transmit antenna, $j = 1, 2, \cdots, M_t$, to the ith receive antenna, $i = 1, 2, \cdots, N_r$. The channel gains are assumed to be independent circularly symmetric complex Gaussian (CSCG) random variables with unit variance, i.e., $\mathcal{CN}(0, 1)$. The received vectors $\mathbf{y}_r \in \mathbb{C}^{N_r \times 1}$ at receiver r, $r = 1, 2$ are given by

$$\mathbf{y}_1 = \mathbf{H}_{11}\,\mathbf{s}_1 + \mathbf{H}_{12}\,\mathbf{s}_2 + \mathbf{n}_1, \quad (1)$$

$$\mathbf{y}_2 = \mathbf{H}_{21}\,\mathbf{s}_1 + \mathbf{H}_{22}\,\mathbf{s}_2 + \mathbf{n}_2, \quad (2)$$

where $\mathbf{s}_t \in \mathbb{C}^{M_t \times 1}$ is the transmitted vector by transmitter t and $\mathbf{S}_t = \mathbb{E}[\mathbf{s}_t\,\mathbf{s}_t^H]$, $t = 1, 2$. The vector $\mathbf{n}_r \in \mathbb{C}^{N_r \times 1}$ is a CSCG random vector with zero mean and identity covariance matrix. Transmitter t is subject to a separate power constraint P_t: $\mathrm{Tr}(\mathbf{S}_t) \leq P_t$. The total power transmitted by both the transmitters is P_T, i.e., $P_T = P_1 + P_2$.

We assume perfect knowledge of all the channel matrices $\mathbf{H}_{rt}$, $r, t = 1, 2$, at both transmitters and at both receivers. The two transmitters do not cooperate, which implies that non-causal knowledge of the other transmitter's data is not available. Similarly, the receivers do not cooperate.

Notation Here, we review some notation which will be needed in section III. The $\mathrm{vec}(\cdot)$ operator stacks the columns of the input matrix sequentially into one column-vector. The Kronecker product between two matrices $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$. Let $\mathbf{A} \in \mathbb{C}^{N \times Q}$. The commutation matrix $\mathbf{K}_{N,Q}$ is a permutation matrix of size $NQ \times NQ$, and it gives the connection between $\mathrm{vec}(\mathbf{A})$ and $\mathrm{vec}(\mathbf{A}^T)$: $\mathbf{K}_{N,Q}\,\mathrm{vec}(\mathbf{A}) = \mathrm{vec}(\mathbf{A}^T)$.

III. MIMO MAC Sum-rate Upper Bound

Let C_X denote the sum-rate capacity of the MIMO X channel. Consider a system where both the receivers cooperate to form a corresponding MIMO MAC channel with the same individual power constraint at the transmitters. Let C_{MAC} be the sum-rate capacity of this MIMO MAC channel. It is clear that $C_X \leq C_{MAC}$. The above outer bound is in general loose. It can be further tightened by assuming noise correlation at both the receivers.

Note that the capacity region of the X channel depends only on the marginal transition probabilities of the channel (i.e., $\mathrm{p}(\mathbf{y}_i|\,\mathbf{s}_i)$) and not on the joint distribution $\mathrm{p}(\mathbf{y}_1, \mathbf{y}_2|\,(\mathbf{s}_1, \mathbf{s}_2))$. Hence, correlation between the noise vectors at both the receivers of the MIMO XC does not affect the MIMO XC capacity region. However, it does affect the sum-rate capacity of the MIMO MAC system, which continues to be an upper bound on the sum-rate capacity of the MIMO XC. Thus, we have $\mathbb{E}[\mathbf{n}_i\mathbf{n}_i^H] = \mathbf{I}, i = 1, 2$, (i.e., at a single receiver, the noise components at different antennas are uncorrelated). Let $\mathbb{E}[\mathbf{n}_1\mathbf{n}_2^H] \triangleq \tilde{\mathbf{X}} \prec \mathbf{I}$. Noise correlation between the multiple antennas within a single receiver affects the capacity of the MIMO XC, and hence is not considered.

Let $\mathbf{z} = [\mathbf{n}_1^T\mathbf{n}_2^T]^T$ be the noise vector in the MAC and let $\mathbf{Z} = \mathbb{E}[\mathbf{z}\mathbf{z}^H]$ denote the noise covariance matrix. Define $\mathbb{S}$ to denote the set of all positive semidefinite noise covariance matrices satisfying the MAC upper bound conditions

$$\mathbb{S} = \left\{\mathbf{Z} : \mathbf{Z} \geq 0, \mathbf{Z} = \begin{bmatrix} \mathbf{I}_{N_1} & \tilde{\mathbf{X}} \\ \tilde{\mathbf{X}}^H & \mathbf{I}_{N_2} \end{bmatrix}\right\}. \quad (3)$$

Thus, for any $\mathbf{Z} \in \mathbb{S}$, the MIMO MAC system sum capacity C_{MAC} is still an upper bound to C_X. An upper bound on C_X can be obtained by

$$C_X \leq \inf_{\mathbf{Z} \in \mathbb{S}} C_{MAC}. \quad (4)$$

[1]We use the following notation: Vectors are denoted by boldface lowercase letters, and matrices are denoted by boldface uppercase letters. $[\cdot]^T$ denotes the transpose operation, $[\cdot]^H$ denotes the Hermitian operation, $\mathrm{Tr}(\cdot)$ denotes the trace operation, and $\mathbb{E}\{\cdot\}$ denotes the expectation operation. $\mathbf{I}_n$ denotes $n \times n$ identity matrix. $\mathbf{A} \geq \mathbf{B}$ implies $\mathbf{A} - \mathbf{B}$ is positive semidefinite.

Let $\mathbf{H}_1^T = [\mathbf{H}_{11}^T \mathbf{H}_{21}^T]$ denote the channel from transmitter 1 to the receiver in the MAC. Similarly, $\mathbf{H}_2^T = [\mathbf{H}_{12}^T \mathbf{H}_{22}^T]$. Let $\mathbf{C}_1 = \mathrm{diag}(\mathbf{I}_{N_1}, \mathbf{0}_{N_2 \times N_2})$ and $\mathbf{C}_2 = \mathrm{diag}(\mathbf{0}_{N_1 \times N_1}, \mathbf{I}_{N_2})$. The constraint that $\mathbf{Z} \in \mathbb{S}$ can be expressed as $\sum_{i=1}^{2} \mathbf{C}_i \mathbf{Z} \mathbf{C}_i = \mathbf{I}_N$ with $\mathbf{Z} \geq 0$. The MAC upper bound (4) can be written as a min-max problem

$$\begin{aligned} C_{MAC}^{up-bnd} &= \min_{\mathbf{Z}} \max_{\mathbf{S}_1, \mathbf{S}_2} \log \frac{|\mathbf{H}_1 \mathbf{S}_1 \mathbf{H}_1^H + \mathbf{H}_2 \mathbf{S}_2 \mathbf{H}_2^H + \mathbf{Z}|}{|\mathbf{Z}|} \\ \text{s.t} \quad & \mathbf{S}_i \geq 0,\ \mathrm{Tr}(\mathbf{S}_i) \leq P_i,\ i = 1, 2 \\ & \mathbf{Z} \geq 0 \\ & \mathbf{C}_1 \mathbf{Z} \mathbf{C}_1 + \mathbf{C}_2 \mathbf{Z} \mathbf{C}_2 = \mathbf{I}_N, \end{aligned} \tag{5}$$

where the maximization is over the set of input covariance matrices $\mathbf{S}_i$ at transmitter i, and the minimization is over all possible noise correlations. The computation of $\mathbf{Z}$ above is not necessarily easy, even though the objective function in (5) is convex in $\mathbf{Z}$. Observe that (5) is similar in form to the sum-rate capacity problem of the broadcast channel which can be written as a minimax problem [4,5]. To our knowledge, we show below that the solution approaches in [4,5,20] for the sum capacity problem of BC cannot be used here, thus justifying the use of primal-dual interior point method to solve (5).

In [4], the sum capacity BC minimax problem is transformed into a single convex minimization problem, which in turn can be solved by standard tools. See also [9] for examples of this conversion for several multiuser and multiple antenna channels. We now attempt to apply this technique to (5). To this end, we rewrite the inner maximization of (5) w.r.t $\mathbf{S}_1, \mathbf{S}_2$ as

$$\begin{aligned} & \min_{\mathbf{X}, \mathbf{S}_1, \mathbf{S}_2} -\log |\mathbf{X}| \\ \text{s.t} \quad & \mathbf{X} = \mathbf{Z}^{-1/2} (\mathbf{H}_1 \mathbf{S}_1 \mathbf{H}_1^H + \mathbf{H}_2 \mathbf{S}_2 \mathbf{H}_2^H) \mathbf{Z}^{-1/2} + \mathbf{I}, \\ & \mathbf{S}_i \geq 0,\ \mathrm{Tr}(\mathbf{S}_i) \leq P_i,\ i = 1, 2. \end{aligned} \tag{6}$$

Using the dual variables $\mathbf{T}, \lambda_1, \lambda_2$, the Lagrangian for this problem can be written as

$$\begin{aligned} \mathcal{L}(\mathbf{X}, \mathbf{S}_1, \mathbf{S}_2, \mathbf{T}, \lambda_1, \lambda_2) = -\log |\mathbf{X}| + \sum_{i=1}^{2} \lambda_i (\mathrm{Tr}(\mathbf{S}_i) - P_i) \\ + \mathrm{Tr}(\mathbf{T}(\mathbf{X} - \sum_{i=1}^{2} \mathbf{Z}^{-1/2} \mathbf{H}_i \mathbf{S}_i \mathbf{H}_i^H \mathbf{Z}^{-1/2} - \mathbf{I})) + \sum_{i=1}^{2} \mathrm{Tr}(\mathbf{V}_i \mathbf{S}_i)) \end{aligned}$$

We differentiate the Lagrangian w.r.t the primal variables $\mathbf{X}, \mathbf{S}_i$ to write the optimality conditions as

$$\mathbf{X}^{-1} = \mathbf{T}, \tag{7}$$

$$\lambda_i \mathbf{I} + \mathbf{V}_i = \mathbf{H}_i^H \mathbf{Z}^{-1/2} \mathbf{T} \mathbf{Z}^{-1/2} \mathbf{H}_i, \quad i = 1, 2. \tag{8}$$

Substituting (7) and (8) in the Lagrangian above, and maximizing it w.r.t the dual variables, the dual problem to (6) can be written as

$$\begin{aligned} & \max_{\mathbf{T}, \lambda_1, \lambda_2} \log |\mathbf{T}| - \mathrm{Tr}(\mathbf{T}) - \lambda_1 P_1 - \lambda_2 P_2 + N \\ \text{s.t} \quad & \mathbf{T} \geq 0,\ \lambda_i \geq 0,\ i = 1, 2, \\ & \mathbf{H}_i^H \mathbf{Z}^{-1/2} \mathbf{T} \mathbf{Z}^{-1/2} \mathbf{H}_i \geq \lambda_i \mathbf{I}, \quad i = 1, 2. \end{aligned} \tag{9}$$

Using (9) in place of the inner maximization in (5), we get

$$\begin{aligned} & \min_{\mathbf{Z}} \min_{\mathbf{T}, \lambda_1, \lambda_2} -\log |\mathbf{T}| + \mathrm{Tr}(\mathbf{T}) + \lambda_1 P_1 + \lambda_2 P_2 - N \\ \text{s.t} \quad & \mathbf{T} \geq 0,\ \mathbf{Z} \in \mathbb{S},\ \lambda_i \geq 0,\ i = 1, 2, \\ & \mathbf{H}_i^H \mathbf{Z}^{-1/2} \mathbf{T} \mathbf{Z}^{-1/2} \mathbf{H}_i \geq \lambda_i \mathbf{I}, \quad i = 1, 2. \end{aligned} \tag{10}$$

Observe that (10) is a non-convex problem. The non-convexity is a result of the coupling of the optimization variables $\mathbf{Z}$ and $\mathbf{T}$ in the second constraint in (10). The non-convexity of the dual problem for the sum capacity of the BC has been tackled in [4] by using uplink-downlink duality. Uplink-downlink duality refers to the fact that under the same sum power constraint, the sum capacity of a MAC and BC are the same [4,5] For point-to-point channels, uplink-downlink duality reduces to the fact that the capacity remains same when the roles of the transmitter and the receiver are reversed under the same power constraint, a fact observed in [1]. Thus, in [4], first the inner maximization of the downlink primal problem is converted into an equivalent uplink problem under the same power constraint. Next, the dual problem to this equivalent uplink problem is derived. This results in a convex constraint of the form $\mathbf{H}_i^H \mathbf{T} \mathbf{H}_i \geq \tilde{\mathbf{Z}}$, instead of the non-convex constraint $\mathbf{H}_i^H \mathbf{Z}^{-1/2} \mathbf{T} \mathbf{Z}^{-1/2} \mathbf{H}_i \geq \lambda \mathbf{I}$, where $\tilde{\mathbf{Z}} = \lambda \mathbf{Z}$.

We could possibly circumvent the non-convexity of (10) by using this approach. We show below that this does not lead to a single convex minimization problem. To this end, consider the inner maximization of (5) w.r.t $\mathbf{S}_1, \mathbf{S}_2$. For a fixed $\mathbf{Z}$, this represents the sum capacity of a MAC channel with individual power constraints P_1, P_2, and channel matrices $\mathbf{H}_i$ replaced by $\mathbf{Z}^{-1/2} \mathbf{H}_i$. In order to use uplink-downlink duality, we need to consider the downlink BC corresponding to this MAC channel. However, uplink-downlink duality refers to the fact that under the same *sum power constraint*, the sum capacity of a MAC and BC are the same [4,5]. Thus, there is no equivalent downlink BC with the same sum capacity as the above MAC channel with individual power constraints. To utilize uplink-downlink duality, we first relax the individual power constraints on $\mathrm{Tr}(\mathbf{S}_i)$ to a sum power constraint $\mathrm{Tr}(\mathbf{S}_1 + \mathbf{S}_1) \leq P_T$. Notice that this leads to a bound which is loose compared to the solution to (5), since a sum power constraint results in a higher capacity. Let $\tilde{\mathbf{H}} = [\mathbf{H}_1 \mathbf{H}_2]$. Using uplink-downlink duality, the MAC sum capacity with channel matrices $\mathbf{Z}^{-1/2} \mathbf{H}_i$ and a sum power constraint can be written in terms of sum capacity of the BC with the channel matrix $\tilde{\mathbf{H}}^H \mathbf{Z}^{-1/2}$ as

$$\begin{aligned} & \min_{\mathbf{B}} \max_{\mathbf{W}} \log |\mathbf{B}^{-1/2} \tilde{\mathbf{H}}^H \mathbf{Z}^{-1/2} \mathbf{W} \mathbf{Z}^{-1/2} \tilde{\mathbf{H}} \mathbf{B}^{-1/2} + \mathbf{I}| \\ \text{s.t} \quad & \mathbf{W} \geq 0,\ \mathrm{Tr}(\mathbf{W}) \leq P_T,\ \mathbf{B} \in \mathbb{S}^1. \end{aligned} \tag{11}$$

where $\mathbf{B}, \mathbf{W}$ are the noise and input covariances in the corresponding BC and the set $\mathbb{S}^1$ is similar to $\mathbb{S}$, except that the diagonal identity matrices are $\mathbf{I}_{M_i}$. It can be observed that whereas the original sum power MAC problem was a single convex maximization problem, (11) is a minimax problem. Using steps similar to (6)-(10), the dual problem to (5) utilizing the uplink-downlink duality can be written as

$$\begin{aligned}
&\min_{\tilde{\mathbf{Z}}}\ \min_{\mathbf{B}}\ \min_{\mathbf{U},\lambda}\ \ -\log|\mathbf{U}| + \mathrm{Tr}(\mathbf{U}) + \lambda P_T - M \\
\text{s.t}\quad & \mathbf{U} \geq 0,\ \ \lambda \geq 0,\ \ \mathbf{Z} \in \mathbb{S},\ \ \mathbf{B} \in \mathbb{S}^1, \\
& \tilde{\mathbf{H}}\,\mathbf{B}^{-1/2}\,\mathbf{U}\,\mathbf{B}^{-1/2}\,\tilde{\mathbf{H}}^H \ \geq\ \tilde{\mathbf{Z}},
\end{aligned} \tag{12}$$

where $\mathbf{U}, \lambda$ are dual variables and $\tilde{\mathbf{Z}} = \lambda\mathbf{Z}$. Alternatively, if we consider the conjugate transpose channel in (11), then the second constraint in (12) is replaced by the constraint $\tilde{\mathbf{H}}^H\,\mathbf{Z}^{-1/2}\,\mathbf{U}\,\mathbf{Z}^{-1/2}\,\tilde{\mathbf{H}}\ \geq\ \tilde{\mathbf{B}}$ and $\tilde{\mathbf{B}} = \lambda\mathbf{B}$. Thus, in both the formulations of the dual problem (12), either $\mathbf{B}$ and $\mathbf{U}$ are coupled as in (12), or $\mathbf{Z}$ and $\mathbf{U}$ are coupled ultimately resulting in the non-convex optimization problem (12). Hence the use of uplink-downlink duality has not eliminated the non-convexity of the dual problem to (5) as was expected. Thus, we have shown that techniques similar to [4] cannot be applied to (5).

In [5], a different approach to duality is taken, where it is proved that the worst noise matrix for the BC sum capacity problem corresponds to an input cost constraint in the equivalent uplink channel. It is then established that the worst noise matrix for the BC leads to an input cost constraint which decouples the inputs of the equivalent point-to-point uplink channel resulting in a sum power constrained MAC, which proves the duality between the BC and the MAC under a sum power constraint. Interestingly, this result is generalized in [20, Theorem 3], where it is shown that a noise covariance constraint in the MAC can be transformed into an input cost constraint in the BC. Using this idea does not help to simplify (5) as we show below. Consider (11) and write $\mathbf{C} = \mathbf{Z}^{-1/2}\mathbf{W}\mathbf{Z}^{-1/2}$. The constraint $\mathrm{Tr}(\mathbf{W}) \leq P_T$ is replaced by the input cost constraint $\mathrm{Tr}(\mathbf{C}\,\mathbf{Z}) \leq P_T$. Thus, (11) can be rewritten as

$$\begin{aligned}
&\min_{\mathbf{B}}\ \max_{\mathbf{C}}\ \ \log|\mathbf{B}^{-1/2}\,\tilde{\mathbf{H}}^H\,\mathbf{C}\,\tilde{\mathbf{H}}\,\mathbf{B}^{-1/2} + \mathbf{I}| \\
\text{s.t}\quad & \mathbf{W} \geq 0,\ \ \mathrm{Tr}(\mathbf{C}\,\mathbf{Z}) \leq P_T,\ \ \mathbf{B} \in \mathbb{S}^1.
\end{aligned} \tag{13}$$

Notice that the above problem is an application of [20, Theorem 3] to (5) for a fixed noise covariance $\mathbf{Z}$. Writing the dual of (13), and minimizing over $\mathbf{Z}$, leads precisely to the non-convex formulation (12).

Finally, in [20], Lagrangian duality theory is utilized to transform the minimax problem to a single maximization problem corresponding to the sum power MAC problem. The technique used in [20] cannot be applied here as the minimax problem (5) cannot be transformed to a single maximization problem as we show below. Using a set of steps similar to that in [20, Sec. III-B], it can be shown that the application of Lagrangian minimax duality to the minimax problem (5) results in the following dual minimax problem

$$\begin{aligned}
&\min_{\mathbf{\Theta}}\ \max_{\mathbf{\Psi}_1,\mathbf{\Psi}_2}\ \ \log\frac{|\mathbf{G}_1\mathbf{\Psi}_1\mathbf{G}_1^H + \mathbf{G}_2\mathbf{\Psi}_2\mathbf{G}_2^H + \mathbf{\Theta}|}{|\mathbf{\Theta}|} \\
\text{s.t}\quad & \mathbf{\Psi}_1 \geq 0,\ \ \mathbf{\Psi}_2 \geq 0,\ \ \mathrm{Tr}(\mathbf{\Psi}_1 + \mathbf{\Psi}_2) \leq 1, \\
& \mathbf{\Theta} \in \mathbb{S}^2,
\end{aligned} \tag{14}$$

where $\mathbf{G}_1^H = [\mathbf{H}_{11}\mathbf{H}_{12}]$ and $\mathbf{G}_2^H = [\mathbf{H}_{21}\mathbf{H}_{22}]$. $\mathbf{\Psi}_i$, $\mathbf{\Theta}$ are $N_i \times N_i$ and $M \times M$ matrices of dual variables respectively and the set $\mathbb{S}^2$ denotes the following

$$\mathbb{S}^2 = \left\{\mathbf{\Theta} : \mathbf{\Theta} \geq 0, \mathbf{\Theta} = \begin{bmatrix} \lambda_1\mathbf{I}_{M_1} & \widehat{\mathbf{X}} \\ \widehat{\mathbf{X}}^H & \lambda_2\mathbf{I}_{M_2} \end{bmatrix}, \lambda_i \geq 0, \sum_{i=1}^{2}\lambda_i P_i \leq 1\right\}.$$

Quite interestingly, (14) can be recognized as the sum power constrained MAC channel obtained by reversing the role of transmitters and receivers in the MIMO X channel and assuming cooperation among the new receivers resulting in a M antenna receiver. Also note that the noise covariances at the new receiver i is scaled by λ_i. However, in contrast to [20], the form of the problem (14) does not allow any more simplifications and any attempt to simplify (14) either results in yet another minimax problem with comparable complexity or a non-convex problem. Thus we have shown that the solution approaches in [4,5,20] cannot be used to solve (5). We instead formulate it as a convex-concave game and solve it directly without further transformations using interior point methods.

A. *Convex-concave game interpretation*

The minimax problem in (5) can be interpreted as a zero-sum, two-player convex-concave game. Let the objective function in (5) be denoted by $f(\mathbf{Z}, \mathbf{S})$, where $\mathbf{S} = \mathrm{diag}(\mathbf{S1}, \mathbf{S2})$. $f(\mathbf{Z}, \mathbf{S})$ is called the payoff function in game theory literature [23]. Player 1 chooses a noise matrix $\mathbf{Z}$ and and player 2 chooses an input covariance matrix $\mathbf{S}$. Based on these choices, player 1 makes a payment to player 2, in the amount $f(\mathbf{Z}, \mathbf{S})$. The goal of player 1 is to minimize this payment, whereas the goal of player 2 is to maximize it. Note that the game is convex-concave, since for each $\mathbf{S}$, $f(\mathbf{Z}, \mathbf{S})$ is a convex function of $\mathbf{Z}$ and for each $\mathbf{Z}$, $f(\mathbf{Z}, \mathbf{S})$ is a concave function of $\mathbf{S}$. Since the payoff function is convex-concave and the constraints in (5) are convex, a saddle point exists. This follows from a minimax theorem in game theory [27]. We say that $\mathbf{Z}^*$, $\mathbf{S}^*$ is a *solution* of the game, or a *saddle-point* (Nash-equilibrium), if for all $\mathbf{Z}$, $\mathbf{S}$,

$$f(\mathbf{Z}^*,\ \mathbf{S})\ \leq\ f(\mathbf{Z}^*,\ \mathbf{S}^*)\ \leq\ f(\mathbf{Z},\ \mathbf{S}^*).$$

Further, the distributions of both $\mathbf{Z}^*$, $\mathbf{S}^*$ are Gaussian [25]. For a general function $g(x, y)$, it always holds that

$$\min_x\ \max_y\ g(x, y)\ \geq\ \max_y\ \min_x\ g(x, y). \tag{15}$$

However, when the saddle point exists, max-min equals min-max, and the MAC upper bound can be formulated as in (5).

Note that, for $\mathbf{Z} = \mathbf{Z}^*$, $\mathbf{S}^*$ maximizes $f(\mathbf{Z}^*,\ \mathbf{S})$, and for $\mathbf{S} = \mathbf{S}^*$, $\mathbf{Z}^*$ minimizes $f(\mathbf{Z},\ \mathbf{S}^*)$. Thus, at the saddle point, neither player can do better by changing his strategy. When f is differentiable and convex-concave, the Karush-Kuhn-Tucker (KKT) conditions are a sufficient and necessary condition for the optimality of the saddle point. This can be proved as follows: since f is a convex function of $\mathbf{Z}$, the optimality condition for $\mathbf{Z}^*$ to be a minimum for $f(\mathbf{Z},\ \mathbf{S}^*)$ is $\nabla_{\mathbf{Z}} f(\mathbf{Z}^*,\ \mathbf{S}^*) = 0$, which is the KKT condition w.r.t $\mathbf{Z}$. Similar argument applies to $\mathbf{S}$.

For a game with a twice-differentiable payoff function, with inequality constraints on $\mathbf{Z}$, $\mathbf{S}$, the solution of the game (5) can be computed using a primal-dual interior point method which simultaneously solves the minimization and maximization parts of (5).

We observe that strong duality holds in case of (5). This follows from Slater's theorem, which states that strong duality holds if Slater's constraint qualifications hold and the problem is convex [23,24]. A generalized version of Slater's condition for the problem (5) is that, there exist $\mathbf{Z}$, $\mathbf{S}_i$ satisfying the following conditions [23]

$$\begin{aligned} \mathbf{S}_i &> 0, \quad \mathrm{Tr}(\mathbf{S}_i) \leq P_i, \quad i = 1,2 \\ \mathbf{Z} &> 0, \quad \mathbf{C}_1\mathbf{Z}\mathbf{C}_1 + \mathbf{C}_2\mathbf{Z}\mathbf{C}_2 = \mathbf{I}_N. \end{aligned} \tag{16}$$

Since $\mathbf{Z}$, $\mathbf{S}_i$ exist satisfying (16), this implies strong duality and also that the dual optimum is attained.

B. Derivation of Primal-Dual Interior Point Method

Primal-dual interior point methods are a class of interior point methods which simultaneously solve both the primal and dual problem. They solve an optimization problem with linear equality and inequality constraints by reducing it to a sequence of linear equality constrained problems. In the following, we derive the interior point method for (5).

First, we write the objective function in (5) as $f_0(\mathbf{Z},\, \mathbf{S}_1,\, \mathbf{S}_2)$. Forming the Lagrangian for (5), we have

$$\begin{aligned} &f_0(\mathbf{Z}, \mathbf{S}_1, \mathbf{S}_2) + \sum_{i=1}^{2} \lambda_i(\mathrm{Tr}(\mathbf{S}_i) - P_i) + \sum_{i=1}^{2} \mathrm{Tr}(\mathbf{\Phi}_i(-\mathbf{S}_i)) \\ &+ \mathrm{Tr}(\mathbf{\Sigma}(\mathbf{C}_1\mathbf{Z}\mathbf{C}_1 + \mathbf{C}_2\mathbf{Z}\mathbf{C}_2 - \mathbf{I}_N)) + \mathrm{Tr}(\mathbf{\Gamma}(-\mathbf{Z})), \end{aligned}$$

where λ_i is the dual variable associated with the power constraint P_i, $i = 1,2$. $\mathbf{\Phi}_i$, $\mathbf{\Gamma}$ are matrices of dual variables associated with the semidefinite constraint on $\mathbf{S}_i$, $\mathbf{Z}$, respectively, and $\mathbf{\Sigma}$ is a block diagonal matrix of dual variables associated with the equality constraint on $\mathbf{Z}$. The starting point for the derivation of the interior point method is the modified KKT conditions for (5). For $i = 1,2$, we have

$$\begin{aligned} \mathbf{Z}^* \geq 0,\ \mathbf{S}_i^* &\geq 0, && (17) \\ \mathbf{C}_1\mathbf{Z}^*\mathbf{C}_1 + \mathbf{C}_2\mathbf{Z}^*\mathbf{C}_2 &= \mathbf{I}_N, && (18) \\ \lambda_i^* \geq 0,\ \mathbf{\Gamma}^* \geq 0,\ \mathbf{\Phi}_i^* &\geq 0, && (19) \\ \mathbf{Z}^*\mathbf{\Gamma}^* = \mathbf{\Gamma}^*\mathbf{Z}^* &= \mathbf{0}, && (20) \\ \mathbf{S}_i^*\mathbf{\Phi}_i^* = \mathbf{\Phi}_i^*\mathbf{S}_i^* &= \mathbf{0}, && (21) \\ -\lambda_i^*(\mathrm{Tr}(\mathbf{S}_i^*) - P_i) &= 1/a, && (22) \\ \mathbf{C}_1\mathbf{\Sigma}^*\mathbf{C}_2 + \mathbf{C}_2\mathbf{\Sigma}^*\mathbf{C}_1 &= \mathbf{0}, && (23) \\ -\nabla_{\mathbf{S}_i} f_0(\mathbf{Z}^*, \mathbf{S}_1^*, \mathbf{S}_2^*) + \lambda_i^*\mathbf{I} - \mathbf{\Phi_i}^{*T} &= 0, && (24) \\ \nabla_{\mathbf{Z}} f_0(\mathbf{Z}^*, \mathbf{S}_1^*, \mathbf{S}_2^*) + \sum_{i=1}^{2} \mathbf{C}_i\mathbf{\Sigma}^{*T}\mathbf{C}_i - \mathbf{\Gamma}^{*T} &= 0, && (25) \end{aligned}$$

where $a > 0$ and $(\cdot)^*$ indicates the optimal value of the variable at the saddle point. Since strong duality holds in case of (5), (20)-(22) represent the modified complementary slackness conditions. (23) is a result of the block diagonal constraint on $\mathbf{\Sigma}$. To find the saddle point, we need to simultaneously solve the system of equations (17)-(25). For $i = 1,2$, let

$$\begin{aligned} \mathbf{Z}^* = \mathbf{Z} + \Delta\mathbf{Z}, \quad \mathbf{S}_i^* &= \mathbf{S}_i + \Delta\mathbf{S}_i, \quad \lambda_i^* = \lambda_i + \Delta\lambda_i, && (26) \\ \mathbf{\Gamma}^* = \mathbf{\Gamma} + \Delta\mathbf{\Gamma}, \quad \mathbf{\Phi}_i^* &= \mathbf{\Phi}_i + \Delta\mathbf{\Phi}_i, \quad \mathbf{\Sigma}^* = \mathbf{\Sigma} + \Delta\mathbf{\Sigma}, && (27) \end{aligned}$$

where $\Delta\mathbf{S}_i, \Delta\mathbf{Z}$ are the primal search directions, and $\Delta\lambda_i$, $\Delta\mathbf{\Phi}_i$, $\Delta\mathbf{\Gamma}$ and $\Delta\mathbf{\Sigma}$ are the dual search directions. In (19), we add both $\mathbf{Z}^*\mathbf{\Gamma}^* = \mathbf{\Gamma}^*\mathbf{Z}^* = \mathbf{0}$ so that the primal and dual search directions $\Delta\mathbf{Z}$, $\Delta\mathbf{\Gamma}$ are Hermitian matrices. Otherwise, though $\mathbf{Z}^*$, $\mathbf{\Gamma}^*$ satisfy equations (17)–(25), they might not be feasible (Hermitian). A similar argument applies to (20).

Using (26) and (27) in (17)–(25), and the first order Taylor's approximation for the gradient of f_0 w.r.t the primal variables $\mathbf{Z}$, $\mathbf{S}_1$ and $\mathbf{S}_2$ (See Appendix) and the vec operator, (17)–(25) can be rewritten as the following system of matrix equations:

$$\begin{aligned} \mathbf{A}_{\mathbf{S}_i}\Delta\mathbf{x}_{\mathbf{S}_i} &= \mathbf{b}_{\mathbf{S}_i}, \quad i = 1,2 && (28) \\ \mathbf{A}_{\mathbf{Z}}\Delta\mathbf{x}_{\mathbf{Z}} &= \mathbf{b}_{\mathbf{Z}}, && (29) \end{aligned}$$

where

$$\mathbf{A}_{\mathbf{S}_i} = \begin{bmatrix} -\nabla^2_{\mathbf{S}_i} f_0 & -\mathbf{K}_{M_i \times M_i} & \mathrm{vec}(\mathbf{I}_{M_i}) \\ -\lambda_i\,\mathrm{vec}(\mathbf{I}_{M_i})^T & \mathbf{0} & P_i - \mathrm{Tr}(\mathbf{S}_i) \\ \mathbf{I}_{M_i} \otimes \mathbf{\Phi}_i & \mathbf{S}_i^T \otimes \mathbf{I}_{M_i} & \mathbf{0} \\ \mathbf{\Phi}_i^T \otimes \mathbf{I}_{M_i} & \mathbf{I}_{M_i} \otimes \mathbf{S}_i & \mathbf{0} \end{bmatrix},$$

$$\mathbf{b}_{\mathbf{S}_i} = -\begin{bmatrix} \mathrm{vec}(\nabla_{\mathbf{S}_i} f_0 - \lambda_i\mathbf{I}_{M_i} + \mathbf{\Phi}_i^T) \\ \lambda_i(P_i - \mathrm{Tr}(\mathbf{S}_i)) - 1/t \\ \mathrm{vec}(\mathbf{\Phi}_i\mathbf{S}_i) \\ \mathrm{vec}(\mathbf{S}_i\mathbf{\Phi}_i) \end{bmatrix},$$

$$\mathbf{A}_{\mathbf{Z}} = \begin{bmatrix} \nabla^2_{\mathbf{Z}} f_0 & -\mathbf{K}_{N\times N} & \mathbf{K}_{N\times N} \\ \mathbf{D}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_2 \\ \mathbf{I}_N \otimes \mathbf{\Gamma} & \mathbf{Z}^T \otimes \mathbf{I}_N & \mathbf{0} \\ \mathbf{\Gamma}^T \otimes \mathbf{I}_N & \mathbf{I}_N \otimes \mathbf{Z} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{b}_{\mathbf{Z}} = -\begin{bmatrix} \mathrm{vec}(\nabla_{\mathbf{Z}} f_0 + (\mathbf{\Sigma} - \mathbf{\Gamma})^T) \\ \mathbf{D}_1\,\mathrm{vec}(\mathbf{Z}) - \mathrm{vec}(\mathbf{I}_N) \\ \mathbf{D}_2\,\mathrm{vec}(\mathbf{\Sigma}) \\ \mathrm{vec}(\mathbf{\Gamma}\mathbf{Z}) \\ \mathrm{vec}(\mathbf{Z}\mathbf{\Gamma}) \end{bmatrix},$$

and

$$\Delta\mathbf{x}_{\mathbf{S}_i} = \begin{bmatrix} \mathrm{vec}(\Delta\mathbf{S}_i) \\ \mathrm{vec}(\Delta\mathbf{\Phi}_i) \\ \Delta\lambda_i \end{bmatrix}, \qquad \Delta\mathbf{x}_{\mathbf{Z}} = \begin{bmatrix} \mathrm{vec}(\Delta\mathbf{Z}) \\ \mathrm{vec}(\Delta\mathbf{\Gamma}) \\ \mathrm{vec}(\Delta\mathbf{\Sigma}) \end{bmatrix},$$

where $\mathbf{D}_1 = \mathbf{C}_1 \otimes \mathbf{C}_1 + \mathbf{C}_2 \otimes \mathbf{C}_2$, $\mathbf{D}_2 = \mathbf{C}_2 \otimes \mathbf{C}_1 + \mathbf{C}_1 \otimes \mathbf{C}_2$, and for brevity, we write f_0 to denote $f_0(\mathbf{Z}, \mathbf{S}_1, \mathbf{S}_2)$. We use $\nabla_{\mathbf{S}_i} f_0$, $\nabla_{\mathbf{Z}} f_0$ to denote the first-order derivative of f_0 w.r.t $\mathbf{S}_i$, $\mathbf{Z}$ respectively and $\nabla^2_{\mathbf{S}_i} f_0$, $\nabla^2_{\mathbf{Z}} f_0$ to denote the Hessian or the second-order derivative of f_0 w.r.t $\mathbf{S}_i$, $\mathbf{Z}$ respectively. In the Appendix, we give expressions for the first and second-order derivatives of f_0 w.r.t the primal variables.

C. Primal-dual interior point algorithm

In this subsection, we describe the algorithm used to solve (5) in Algorithm 1. Let $\mathbf{x}_{\mathbf{Z}} = \mathrm{vec}([\mathbf{Z} \quad \mathbf{\Gamma} \quad \mathbf{\Sigma}])$ and $\mathbf{x}_{\mathbf{S}_i} = \mathrm{vec}([\mathbf{S}_i \quad \mathbf{\Phi}_i \quad \lambda_i])$, $i = 1,2$. μ, ϵ are parameters of the algorithm and m denotes the number of modified complementary slackness conditions, where $m = 2$ from (22). The updated primal and dual variables in the kth iteration of the algorithm do not satisfy the KKT conditions (17)–(25), except in the limit as the algorithm converges.

Algorithm 1. Primal-dual interior point method

1) *Initialize* $\mathbf{Z} > 0$, $\boldsymbol{\Gamma} > 0$, $\boldsymbol{\Sigma} = 0$, $\mathbf{S}_i > 0$, $\boldsymbol{\Phi}_i > 0$, $\lambda_i > 0$, $i = 1, 2$, $\mu > 1$, $\epsilon > 0$.
2) *Evaluate* $a = \mu m/\hat{\eta}$.
3) *Compute primal-dual search directions*, $\Delta \mathbf{x}_{\mathbf{Z}}$, $\Delta \mathbf{x}_{\mathbf{S}_i}$, $i = 1, 2$, *using* (28)-(29).
4) *Line search and update*: Determine step length $v > 0$, $u_i > 0$ and set $\mathbf{x}_{\mathbf{Z}} = \mathbf{x}_{\mathbf{Z}} + v\Delta\mathbf{x}_{\mathbf{Z}}$, $\mathbf{x}_{\mathbf{S}_i} = \mathbf{x}_{\mathbf{S}_i} + u_i \Delta \mathbf{x}_{\mathbf{S}_i}$.
5) *Compute primal and dual residuals*: $\mathbf{R}^{dual}_{\mathbf{S}_i}$, $\mathbf{R}^{pri}_{\mathbf{Z}}$, $\mathbf{R}^{dual}_{\mathbf{Z}}$ and *surrogate duality gap* $\hat{\eta}$.
6) *If* $\|\mathbf{R}^{dual}_{\mathbf{S}_i}\|_F \leq \epsilon$, $\|\mathbf{R}^{pri}_{\mathbf{Z}}\|_F \leq \epsilon$, $\|\mathbf{R}^{dual}_{\mathbf{Z}}\|_F \leq \epsilon$ and $\hat{\eta} \leq \epsilon$, *stop. Otherwise goto step 2.*

Hence, we define the primal and dual residuals w.r.t $\mathbf{Z}$ and $\mathbf{S}_i$, $i = 1, 2$ at the kth iteration as:

$$\begin{aligned} \mathbf{R}^{dual}_{\mathbf{S}_i} &= \nabla_{\mathbf{S}_i} f_0 - \lambda_i \mathbf{I}_{M_i} + \boldsymbol{\Phi}_i^T, \ i = 1, 2, \\ \mathbf{R}^{dual}_{\mathbf{Z}} &= \nabla_{\mathbf{Z}} f_0 + (\boldsymbol{\Sigma} - \boldsymbol{\Gamma})^T, \\ \mathbf{R}^{pri}_{\mathbf{Z}} &= \mathbf{C}_1 \mathbf{Z} \mathbf{C}_1 + \mathbf{C}_2 \mathbf{Z} \mathbf{C}_2 - \mathbf{I}_N. \end{aligned} \tag{30}$$

The parameter $\hat{\eta}$ is called surrogate duality gap. This would be the duality gap if the primal and dual residuals in (30) were equal to zero. It is given by

$$\hat{\eta} = \sum_{i=1}^{2} \left(\lambda_i (P_i - \mathrm{Tr}(\mathbf{S}_i)) + \mathrm{Tr}(\boldsymbol{\Phi}_i \mathbf{S}_i \mathbf{S}_i^T \boldsymbol{\Phi}_i^T) \right) + \mathrm{Tr}(\boldsymbol{\Gamma} \mathbf{Z} \mathbf{Z}^T \boldsymbol{\Gamma}^T).$$

It is not hard to see that the value of a that corresponds to the surrogate duality gap $\hat{\eta}$ is $m/\hat{\eta}$. The line search in Algorithm 1 is a standard backtracking line search, based on the norm of the primal and dual residuals, modified to ensure that $\mathbf{Z} > 0$, $\mathbf{S}_i > 0$, $\boldsymbol{\Phi}_i > 0$, $\boldsymbol{\Gamma} > 0$ and $\lambda_i > 0$ for $i = 1, 2$ [23]. The convergence of Algorithm 1 follows from the convergence of the primal-dual interior point method [23].

IV. Modified MMK Scheme

In this section, we propose a scheme similar to the MMK scheme [12], with the objective of maximizing the sum-rate of the MIMO XC, which employs DPC at the transmitters and successive decoding at the receivers. Further, we optimize over the encoding and decoding order at the transmitters and receivers. In the MMK scheme, the authors employ zero forcing DPC (ZF-DPC), at the transmitters and successive decoding along with whitening filters at the receivers to decompose the system into four parallel channels. The filters are designed to exploit the structure of the channel matrices to achieve the highest multiplexing gain.

We describe the proposed modified MMK (m-MMK) scheme below. Since the DPC at the transmitter results in independent streams for both the receivers, we write $\mathbf{s}_1 = \mathbf{s}_{11} + \mathbf{s}_{21}$, and $\mathbf{s}_2 = \mathbf{s}_{12} + \mathbf{s}_{22}$, where $\mathbf{s}_{rt}$ indicates a transmission of stream from transmitter t to receiver r. Let the encoding order at transmitter i be π_{ti}, $i = 1, 2$, and the decoding order at receiver j be π_{rj}, $j = 1, 2$.

Instead of explaining the general case, we discuss an illustrative example. Let $\pi_{ti} = (1, 2)$ at transmitter $i = 1, 2$ and $\pi_{rj} = (1, 2)$ at receiver $j = 1, 2$. This means that $\mathbf{s}_{11}$ is encoded first at transmitter 1 and $\mathbf{s}_{21}$ is encoded with full non-causal knowledge of $\mathbf{s}_{11}$. Thus, at receiver 2, $\mathbf{s}_{11}$ does not cause interference while decoding $\mathbf{s}_{21}$. Similarly, $\mathbf{s}_{12}$ does not interfere with $\mathbf{s}_{22}$. However, $\mathbf{s}_{21}$ and $\mathbf{s}_{22}$ interfere with decoding of $\mathbf{s}_{11}$ and $\mathbf{s}_{12}$ at receiver 1. Successive decoding is employed at both receivers to decode the intended signals. The decoding order at receiver 1, $\pi_{r1} = (1, 2)$ means that $\mathbf{s}_{11}$ is decoded and canceled out before decoding $\mathbf{s}_{12}$. Similarly, $\pi_{r2} = (1, 2)$ indicates that $\mathbf{s}_{21}$ is decoded first and canceled before proceeding to decode $\mathbf{s}_{22}$. We use $\mathbf{N}_i$ to denote the noise plus interference from unintended signals at receiver i. For the encoding order above, $\mathbf{N}_1 = \mathbf{H}_{11}\mathbf{S}_{21}\mathbf{H}_{11}^H + \mathbf{H}_{12}\mathbf{S}_{22}\mathbf{H}_{12}^H + \mathbf{I}_{N_1}$ and $\mathbf{N}_2 = \mathbf{I}_{N_2}$. Let R_{rt} denote the rate of stream $\mathbf{s}_{rt}$, $(r, t) = (1, 2)$. Thus, for the above ordering, the following rate vector is achievable

$$R_{rt} \ \leq \ \log \frac{|\mathbf{H}_{rt}\mathbf{S}_{rt}\mathbf{H}_{rt}^H + \mathbf{N}_{rt}|}{|\mathbf{N}_{rt}|}, \quad (r, t) = 1, 2, \tag{31}$$

where $\mathbf{N}_{rt}$ is the effective noise for stream $\mathbf{s}_{rt}$ and for the above encode and decode ordering, $\mathbf{N}_{11} = \mathbf{H}_{12}\mathbf{S}_{12}\mathbf{H}_{12}^H + \mathbf{N}_1$, $\mathbf{N}_{12} = \mathbf{N}_1$, $\mathbf{N}_{21} = \mathbf{H}_{22}\mathbf{S}_{22}\mathbf{H}_{22}^H + \mathbf{N}_2$ and $\mathbf{N}_{22} = \mathbf{N}_2$.

Observe that effective noise $\mathbf{N}_{rt}$, $(r, t) = 1, 2$, is both a function of the ordering at the transmitters/receivers and the input covariance matrices. The apriori knowledge of the effective noise and the decoding order used at the both the receivers is needed at the transmitter to allocate the rates to the two streams appropriately. Since they are apriori not known, we adopt the following iterative algorithm. The achievable sum-rate capacity depends on the ordering used at the transmitters/receivers. Hence, we calculate the sum-rate for all the 16 combinations of ordering and take the maximum.

Algorithm 2. Proposed modified MMK scheme

1) *Assume ordering* π_{ti} *at transmitter* $i = 1, 2$ *and ordering* π_{rj} *at receiver* $j = 1, 2$ *and set* $\mathbf{N}_i = \mathbf{I}$, $i = 1, 2$.
2) *Compute DPC matrices* $\mathbf{S}_{rt}$ *at transmitter* t, $(t, r) = (1, 2)$.
3) *Recompute* $\mathbf{N}_i$, $i = 1, 2$.
4) *Repeat steps 2–3, till the DPC matrices* $\mathbf{S}_{rt}$, $(t, r) = (1, 2)$ *converge to within* $\epsilon \geq 0$.
5) *Evaluate the sum-rate* $R = \sum_{r=1}^{2} \sum_{t=1}^{2} R_{rt}$.
6) *Repeat steps 1–5 for all 16 combinations of ordering*, π_{ti}, π_{rj} $(i, j) = 1, 2$ *and take the maximum over all the sum-rates.*

We use the efficient algorithm in [21] to compute the DPC matrices. For a given ordering at the transmitters/receivers, simulation results show that the algorithm converges fast.

V. Simulation Results

We consider a MIMO X channel with $M_1 = M_2 = 3$ antennas at the transmitters and $N_1 = N_2 = 3$ antennas at the receivers. The maximum multiplexing gain of the MIMO XC, for this configuration of antennas is 4 [10]. The total power is divided equally between the two transmitters, $P_1 = P_2 = P_T/2$. The signal-to-noise ratio (SNR) is defined as P_T/σ_n^2, where σ_n^2 is the variance of the CSCG noise

SNR (dB)	Capacity: bits/s/Hz	
	MAC Upper Bound	MAC Upper Bound: with least favorable noise
0	5.8196	5.3213
5	10.4725	9.9879
10	16.7783	16.4142
15	24.4791	24.2603
20	33.1513	33.0465
25	42.4444	42.4035
30	52.0951	52.0810
35	61.9262	61.9216
40	71.8384	71.8371

TABLE I

PERFORMANCE OF MAC UPPER BOUND WITH LEAST FAVORABLE NOISE

at a receive antenna. We use 5000 realizations of CSCG channels and regard the average sum-rates to evaluate the bounds discussed in Sec. III and IV.

In Table I, we give the average sum-rate of MAC without noise correlation and MAC with least favorable noise correlation. We choose the following values for Algorithm 1: $\mu = 10$ and set the accuracy level $\epsilon = 10^{-6}$. First, it can be observed that the outer bound derived in Sec. III is tighter than the MAC upper bound without noise correlation. However, the difference vanishes in the high SNR regime. This can be attributed to the fact that, at the MAC receiver, in the high SNR region, interference from other transmitters limits capacity and thermal noise plays a much smaller role.

In Fig. 2, we plot the outer bound in Sec. III and the inner bound derived in Sec. IV. We also give two closely related schemes, MMK scheme in [12] and the coding schemes in [13] called MMK Joint Design (MMK-JD) for reference. It is seen that the outer bound is quite close to the achievable region of the proposed m-MMK scheme at low SNRs, and is moderately close in the medium SNR regime. However, the difference between the bounds increases rapidly, once we approach higher SNRs. This can be explained as follows. Consider the outer bound in Sec. III. There are two streams in the MAC as opposed to four in the proposed m-MMK scheme. Notice that all the $N_1 + N_2$ antennas are used to decode both the streams. What is seen as interference in the m-MMK scheme is actually used by the MAC to decode the received streams using all the antennas. Additionally, after one of the streams is decoded and canceled out, it enables the allocation of a much higher rate to the other stream. In contrast, in the m-MMK scheme, once we enter the high SNR region, high interference from unintended signals at the receivers causes severe capacity degradation. Also note that the proposed m-MMK scheme outperforms both MMK and MMK Joint design. This is primarily due to optimization over the transmit/receiver ordering and to a lesser extent on the use of optimal DPC as opposed to ZF-DPC in MMK. In MMK Joint Design, ZF-DPC transmit matrices are used and receive whitening filters are used and successive decoding is not employed. Thus, the almost constant difference between the two MMK schemes can be attributed solely to the use of successive decoding in MMK.

VI. CONCLUSIONS

We investigated the sum-rate capacity of the MIMO X channel. We derived an upper bound on the sum-rate capacity

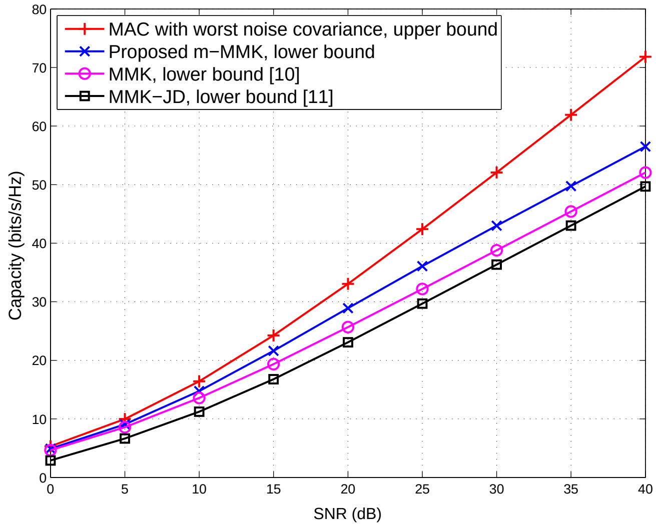

Fig. 2. Comparison of MAC upper bound with least favorable noise correlation and achievable region of the proposed m-MMK scheme.

of the MIMO XC under individual power constraint at each transmitter. We first obtained an upper bound through the MAC sum-rate capacity, assuming receiver cooperation. We then tightened this bound by considering noise correlation between the receivers and derived the worst noise covariance matrix. We formulated this problem as a convex-concave minimax problem and solved it using a primal-dual interior point method which simultaneously solved both the maximization and minimization parts of the problem. We also proposed a scheme similar to that in [12], which employed dirty paper coding at the transmitters and successive decoding at the receivers. We showed that the derived upper bound is close to the achievable region of the proposed scheme at low to medium SNRs.

APPENDIX

The first order Taylor's approximation for $\nabla_{\mathbf{Z}} f_0(\mathbf{Z} + \Delta\mathbf{Z}, \mathbf{S}_1, \mathbf{S}_2)$, when $\mathbf{S}_1$, $\mathbf{S}_2$ are held constant is

$$\begin{aligned}\nabla_{\mathbf{Z}} f_0(\mathbf{Z} + \Delta\mathbf{Z}, \mathbf{S}_1, \mathbf{S}_2) \;&=\; \nabla_{\mathbf{Z}} f_0(\mathbf{Z}, \mathbf{S}_1, \mathbf{S}_2) + \\ &\sum_k \sum_l (\nabla_{\Delta z_{kl}}(\nabla_{\mathbf{Z}} f_0))\Delta z_{kl}.\end{aligned}$$

Using the vec$(\cdot)$ operator, we write the above equation as

$$\mathrm{vec}(\nabla_{\mathbf{Z}} f_0(\mathbf{Z} + \Delta\mathbf{Z}, \mathbf{S}_1, \mathbf{S}_2)) = \mathrm{vec}(\nabla_{\mathbf{Z}} f_0) + \nabla^2_{\mathbf{Z}} f_0 \,\mathrm{vec}(\mathbf{Z}),$$

where for brevity, we write f_0 to denote $f_0(\mathbf{Z}, \mathbf{S}_1, \mathbf{S}_2)$ and $\nabla^2_{\mathbf{Z}} f_0$ is the second-order derivative or Hessian of f_0 w.r.t $\mathbf{Z}$ and is defined below. Similar expressions hold for $\nabla_{\mathbf{S}_i} f_0$ when $\mathbf{S}_i$ alone is varied. Below, we give expressions for the first and second-order derivatives of f_0: $\nabla_{\mathbf{Z}} f_0$, $\nabla_{\mathbf{S}_i} f_0$, $\nabla^2_{\mathbf{Z}} f_0$ and $\nabla^2_{\mathbf{S}_i} f_0$, for $i = 1, 2$:

$$\mathbf{R}_i = \mathbf{H}_i^H(\mathbf{H}_1\mathbf{S}_1\mathbf{H}_1^H + \mathbf{H}_2\mathbf{S}_2\mathbf{H}_2^H + \mathbf{Z})^{-1}\mathbf{H}_i, \quad (32)$$

$$\nabla_{\mathbf{S}_i} f_0 = \mathbf{R}_i^T, \quad (33)$$

$$\mathbf{E} = (\mathbf{Z})^{-1}, \quad (34)$$

$$\mathbf{F} = (\mathbf{H}_1\mathbf{S}_1\mathbf{H}_1^H + \mathbf{H}_2\mathbf{S}_2\mathbf{H}_2^H + \mathbf{Z})^{-1}, \quad (35)$$

$$\nabla_{\mathbf{Z}} f_0 = \mathbf{E}^T - \mathbf{F}^T, \quad (36)$$
$$\nabla^2_{\mathbf{S}_i} f_0 = -\mathbf{K}_{M_i \times M_i}(\mathbf{R}_i^T \otimes \mathbf{R}_i), \quad (37)$$
$$\nabla^2_{\mathbf{Z}} f_0 = -\mathbf{K}_{N \times N}(\mathbf{E}^T \otimes \mathbf{E} - \mathbf{F}^T \otimes \mathbf{F}). \quad (38)$$

(33) and (36) follow from Lemma 1, while (37) and (38) follow from Lemma 2.

Lemma 1. Let $\mathbf{H}, \mathbf{S}$ and $\mathbf{Z} \in \mathbb{C}^{N \times N}$ and let $y = \log|\mathbf{HSH}^H + \mathbf{Z}|$. Then $\nabla_{\mathbf{S}} y = \big(\mathbf{H}^H(\mathbf{HSH}^H + \mathbf{Z})^{-1}\mathbf{H}\big)^T$.

Proof. The above result can be proved by repeatedly applying chain rule for matrix differentials [26]. In Table II, we list some differentials of functions of a complex matrix $\mathbf{X}$ used in the proof [26]. The differential of y, $\mathrm{d}y$ is given by

$$\begin{aligned} \mathrm{d}y &= \mathrm{Tr}\big((\mathbf{HSH}^H + \mathbf{Z})^{-1}\mathrm{d}(\mathbf{HSH}^H + \mathbf{Z})\big) \quad (39)\\ &= \mathrm{Tr}\big((\mathbf{HSH}^H + \mathbf{Z})^{-1}\mathbf{H}(\mathrm{d}\mathbf{S})\mathbf{H}^H\big)\\ &= \mathrm{Tr}\big(\mathbf{H}^H(\mathbf{HSH}^H + \mathbf{Z})^{-1}\mathbf{H}\,\mathrm{d}\mathbf{S}\big). \end{aligned}$$

The final result follows from the relation between the derivative and the differential of a scalar function g of a complex matrix $\mathbf{X}$ [26]: if $\mathrm{d}g = \mathrm{Tr}\big(\mathbf{A}^T\mathrm{d}\mathbf{X}\big)$, then $\nabla_{\mathbf{X}} g = \mathbf{A}$. □

Lemma 2. Let $\mathbf{H}$, $\mathbf{S}$, $\mathbf{Z}$ and y be as in Lemma 1. Then $\nabla^2_{\mathbf{S}} y = -\mathbf{K}_{N,N}\big(\nabla_{\mathbf{S}} y \otimes (\nabla_{\mathbf{S}} y)^T\big)$

Proof. The second differential $\mathrm{d}^2 y$ is given by $\mathrm{d}^2 y = \mathrm{d}(\mathrm{d}y) = \mathrm{d}(\mathrm{d}y)^T$. The first differential $\mathrm{d}y$ in (39) can be alternately written as $\mathrm{d}y = \mathrm{vec}^T(\nabla_{\mathbf{S}} y)\,\mathrm{dvec}(\mathbf{S})$ [26]. Thus,

$$\begin{aligned} \mathrm{d}^2 y &= \mathrm{d}(\mathrm{dvec}^T(\mathbf{S})\ \mathrm{vec}(\nabla_{\mathbf{S}} y)),\\ &= \mathrm{dvec}^T(\mathbf{S})\,\mathrm{dvec}(\nabla_{\mathbf{S}} y). \quad (40) \end{aligned}$$

Let $\mathbf{W} = \big(\nabla_{\mathbf{S}} y\big)^T = \mathbf{H}^H\mathbf{X}^{-1}\mathbf{H}$, where $\mathbf{X} = (\mathbf{HSH}^H + \mathbf{Z})$. Then,

$$\begin{aligned} \mathrm{d}\mathbf{W} &= \mathbf{H}^H\big(\mathrm{d}\mathbf{X}^{-1}\big)\mathbf{H},\\ &= -\mathbf{H}^H\mathbf{X}^{-1}\big(\mathrm{d}\mathbf{X}\big)\mathbf{X}^{-1}\mathbf{H},\\ &= -\mathbf{H}^H\mathbf{X}^{-1}\mathbf{H}\big(\mathrm{d}\mathbf{S}\big)\mathbf{H}^H\mathbf{X}^{-1}\mathbf{H},\\ &= -\mathbf{W}\big(\mathrm{d}\mathbf{S}\big)\mathbf{W}. \quad (41) \end{aligned}$$

Using the following result in matrix theory, $\mathrm{vec}(\mathbf{ACB}) = (\mathbf{B}^T \otimes \mathbf{A})\,\mathrm{vec}(\mathbf{C})$, (41) can be written as

$$\mathrm{dvec}(\mathbf{W}) = -\big(\mathbf{W}^T \otimes \mathbf{W}\big)\mathrm{dvec}(\mathbf{S}),$$

and

$$\mathrm{dvec}(\nabla_{\mathbf{S}} y) = -\mathbf{K}_{N,N}\big(\nabla_{\mathbf{S}} y \otimes (\nabla_{\mathbf{S}} y)^T\big)\,\mathrm{dvec}(\mathbf{S}). \quad (42)$$

Using (42) in (40), we get

$$\mathrm{d}^2 y = -\mathrm{dvec}^T(\mathbf{S})\,\mathbf{K}_{N,N}\big(\nabla_{\mathbf{S}} y \otimes (\nabla_{\mathbf{S}} y)^T\big)\,\mathrm{dvec}(\mathbf{S}). \quad (43)$$

The Hessian of y w.r.t $\mathbf{S}$ can be identified from (43) as [26]

$$\nabla^2_{\mathbf{S}} y = -\mathbf{K}_{N,N}\big(\nabla_{\mathbf{S}} y \otimes (\nabla_{\mathbf{S}} y)^T\big). \quad (44)$$

□

Function	Differential of function
$\mathbf{AXB}$	$\mathbf{A}(\mathrm{d}\mathbf{X})\mathbf{B}$
$\mathbf{X}^{-1}$	- $\mathbf{X}^{-1}(\mathrm{d}\mathbf{X})\mathbf{X}^{-1}$
$\log|\mathbf{X}|$	$\mathrm{Tr}(\mathbf{X}^{-1}\mathrm{d}\mathbf{X})$

TABLE II

LIST OF DIFFERENTIALS OF FUNCTIONS

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