Capacity Bounds for the Gaussian X Channel

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***Abstract*—We consider bounds for the capacity region of the Gaussian X channel (XC), a system consisting of two transmit-receive pairs, where each transmitter communicates with both the receivers. We first classify the XC into two classes, the strong XC and the mixed XC. In the strong XC, either the direct channels are stronger than the cross channels or vice-versa, whereas in the mixed XC, one of the direct channels is stronger than the corresponding cross channel and vice-versa. After this classification, we give outer bounds on the capacity region for each of the two classes. This is based on the idea that when one of the messages is eliminated from the XC, the rate region of the remaining three messages are enlarged. We make use of the Z channel, a system obtained by eliminating one message and its corresponding channel from the X channel, to bound the rate region of the remaining messages. The outer bound to the rate region of the remaining messages defines a subspace in $\mathbb{R}_+^4$ and forms an outer bound to the capacity region of the XC. Thus, the outer bound to the capacity region of the XC is obtained as the intersection of the outer bounds to the four combinations of the rate triplets of the XC. Using these outer bounds on the capacity region of the XC, we derive new sum-rate outer bounds for both strong and mixed Gaussian XCs and compare them with those existing in literature. We show that the sum-rate outer bound for strong XC gives the sum-rate capacity in three out of the four sub-regions of the strong Gaussian XC capacity region. In case of mixed Gaussian XC, we recover the recent results in [11] which showed that the sum-rate capacity is achieved in two out of the three sub-regions of the mixed XC capacity region and give a simple alternate proof of the same.**

keywords: *Capacity region, X channel, interference channel, sum capacity.*

I. Introduction

The capacity of wireless channels has attracted a lot of interest. A major source of performance bottleneck limiting the capacity of wireless systems is caused by interference from the reception of unintended signals at the receivers. A basic model in information theory to study the nature and effect of interference is the the two-user interference channel (IC), consisting of two point-to-point links with additive white Gaussian noise and transmissions on either link interfere with each other. The IC has been the subject of some intense scrutiny by researchers for the past three decades. In spite of this, the capacity region of even the simple two-user Gaussian IC is not known. Recently in [1], some progress has been made in this regard and the capacity of the Gaussian interference channel is characterized to within one bit.

The X channel (XC) is a generalization of the interference channel; there are two transmitter – receiver pairs, and each transmitter intends to communicate with both receivers. It is interesting to note that the multiple access channel (MAC), the broadcast channel (BC), and the IC are contained within the XC and can be obtained as special cases of the XC.

Although the XC is a close cousin of the IC, very little is known regarding the capacity region of the XC. The best known achievable region is due to Koyluoglu, Shahmohammadi, and El Gamal [2]. This rate region when specialized to the IC, was shown to reduce to the Han and Kobayashi rate region [3], which is the best known achievable region for the IC. However, no simplification of Koyluoglu-Shahmohammadi-Gamal rate region was given and its characterization is extremely complicated.

The degrees of freedom of the multiple-input multiple-output (MIMO) X channel is shown to be $\frac{4M}{3}$, with $M > 1$ antennas at each node [4]. It is shown that the concept of interference alignment coupled with zero forcing achieves the highest number of degrees of freedom. It was later shown in [5] that $4/3$ is indeed the degrees of freedom for the $M = 1$ case and introduced the novel idea of asymmetric complex signaling to achieve the outer bound. In [6], the authors combine dirty paper coding, zero forcing and successive decoding methods to obtain signaling schemes which achieve the highest multiplexing gain or the degrees of freedom. They eventually transform the XC into four parallel channels.

The Etkin-Tse-Wang (ETW) sum-rate outer bound [1] derived for the interference channel was extended to the XC in [7]. Also, the sum-rate capacity result for the Gaussian interference channel in the low-interference regime [8]–[10] was extended to the Gaussian X channel. Thus, for a class of channel coefficients, treating interference as noise is sum-rate capacity optimal. In [11], the sum-rate capacity of the XC is obtained for a class of channel coefficients and power levels. When these conditions are met, the sum-rate capacity is shown to be achieved by transmitting only two messages to one of the receivers, i.e., a MAC at either receiver 1 or receiver 2.

In this work, we make progress with regard to the capacity region of the XC. We first classify the XC into two broad classes: *strong* XC and the *mixed* XC. The strong XC corresponds to a class of X channels where either the direct channels are stronger than the corresponding cross channels, or the cross channels are stronger than the corresponding direct channels. In the mixed XC, as the name suggests, one of the direct channels is stronger than the corresponding cross channel, whereas the other cross channel is stronger than the corresponding direct channel or vice-versa. After this classification, we give outer bounds on the capacity region for each of the two classes. This is based on the idea that when one of the messages is eliminated from the XC, the rate region of the remaining three messages are enlarged. We make use of the

Z channel, a system obtained by eliminating one message and its corresponding channel from the X channel, to bound the rate region of the remaining messages. We show that the outer bound to the rate region of the remaining messages defines a subspace in $\mathbb{R}_+^4$ and forms an outer bound to the capacity region of the XC. Thus, the outer bound to the capacity region of the XC is obtained as the intersection of the outer bounds to the four combinations of the rate triplets of the XC. Using these bounds on the capacity region of the XC, we derive new sum-rate outer bounds for both strong and mixed Gaussian XCs and compare them with those existing in literature. We show that the derived sum-rate outer bounds give sum-rate capacity in certain regions of the XC capacity region. We summarize these results below.

1) Strong XC:

- The sum-rate outer bound gives the sum-rate capacity in three out of the four sub-regions of the strong XC capacity region.
- In one of the regions, it is optimal to operate the XC as an IC and treat interference as noise. This corresponds to the noisy-interference or low-interference sum-rate capacity result obtained in [7]. However, we show that the region we obtain is much larger and contains the region in [7] as a subset.
- In the other two regions, it is optimal to operate the XC as a MAC to either receiver 1 or receiver 2.
- We show that the new sum-rate bound outperforms both ETW bounds given in [7].

2) Mixed XC:

- The sum-rate outer bounds give the sum-rate capacity in two out of the three sub-regions of the mixed XC capacity region. This result was first obtained in [11]. We give an alternate proof of this result and show that it arises as a natural consequence of the outer bound to the capacity region.
- In both regions, it is optimal to operate the XC as a MAC to one of the receivers.
- We show that the sum-rate outer bounds outperform the ETW bounds [7] in the above two regions, whereas in the third region, further analysis is needed to ascertain the comparative tightness of the bounds.

The rest of this paper is organized as follows. The system model is presented in Section II. In Section III, we discuss the classification of XCs. In Section IV, we collect some results on the capacity region of the Z channel, and make use of these results to derive an outer bound on the capacity region of the XC in Section V. In Sections VI and VII, we derive the outer bounds on the capacity region of strong and mixed XC, respectively, and give new sum-rate outer bounds for both classes. Conclusions are presented in Section VIII.

We use lowercase letters for scalars and boldface lowercase letters for vectors. $h(\cdot)$ denotes binary differential entropy of a continuous random variable or vector, $I(\cdot;\cdot)$ denotes mutual information, and $E\{\cdot\}$ denotes the expectation operation. All logarithms are to base 2 unless otherwise specified.

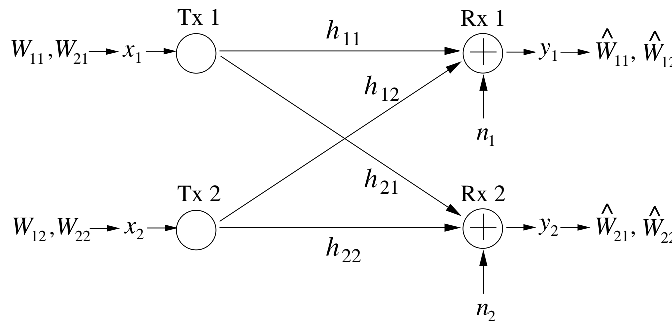

Fig. 1. Gaussian X channel system model.

Class	Name	Channel Constraints
A	Strong Direct Channel Gain XC	$\lvert h_{11}\rvert^2 \geq \lvert h_{21}\rvert^2$; $\lvert h_{22}\rvert^2 \geq \lvert h_{12}\rvert^2$
B	Strong Cross Channel Gain XC	$\lvert h_{11}\rvert^2 \leq \lvert h_{21}\rvert^2$; $\lvert h_{22}\rvert^2 \leq \lvert h_{12}\rvert^2$
C	Mixed Channel Gain XC 1	$\lvert h_{11}\rvert^2 \geq \lvert h_{21}\rvert^2$; $\lvert h_{22}\rvert^2 \leq \lvert h_{12}\rvert^2$
D	Mixed Channel Gain XC 2	$\lvert h_{11}\rvert^2 \leq \lvert h_{21}\rvert^2$; $\lvert h_{22}\rvert^2 \geq \lvert h_{12}\rvert^2$

TABLE I
GENERAL CLASSIFICATION OF X CHANNELS

II. SYSTEM MODEL

The Gaussian X channel system model is shown in Fig. 1. We consider the single-input single-output (SISO) case, where both transmitters and both receivers are equipped with single antenna each. As shown in Fig. 1, the X channel has four independent messages, W_{11}, W_{12}, W_{21}, W_{22}, where W_{ij} is the message transmitted from transmitter j to receiver i. We assume a flat-fading environment. Let h_{rt} denote the channel gain from transmitter t to receiver r, $\forall\ t, r \in \{1, 2\}$. The channel gains are assumed to be independent circularly symmetric complex Gaussian (CSCG) random variables with unit variance, i.e., $h_{rt} \sim \mathcal{CN}(0, 1)$. The received symbols y_r at receiver r, $r = 1, 2$ are given by

$$y_1 = h_{11}\, x_1 + h_{12}\, x_2 + n_1 \quad (1)$$

$$y_2 = h_{21}\, x_1 + h_{22}\, x_2 + n_2, \quad (2)$$

where x_t is the transmitted symbol by transmitter t and n_r is a CSCG random variable with unit variance. Transmitter t is subject to a separate power constraint $\mathbb{E}[|x_t|^2] \leq P_t$.

III. CLASSIFICATION OF X CHANNELS

In this section, we attempt to classify the X channel based on the channel parameters. Depending on the magnitude of the channel parameters, the XC can be classified into the four classes shown in Table I.

In class A, i.e., strong direct channel gain XC, the direct channel gains $|h_{11}|^2$ and $|h_{22}|^2$ are greater than the cross channel gains $|h_{21}|^2$ and $|h_{12}|^2$, respectively. Exactly the opposite is true for class B. As the name suggests, in case of the mixed channel gain XC, one of the direct channels is stronger than the corresponding cross channel, whereas the other cross channel

Name	Channel constraints	
	General XC	Standard form XC
Strong XC	$\lvert h_{11}\rvert^2 \geq \lvert h_{21}\rvert^2$; $\lvert h_{22}\rvert^2 \geq \lvert h_{12}\rvert^2$	$\lvert\alpha\rvert^2 \leq 1$; $\lvert\beta\rvert^2 \leq 1$
	$\lvert h_{11}\rvert^2 \leq \lvert h_{21}\rvert^2$; $\lvert h_{22}\rvert^2 \leq \lvert h_{12}\rvert^2$	$\lvert\alpha\rvert^2 \geq 1$; $\lvert\beta\rvert^2 \geq 1$
Mixed XC	$\lvert h_{11}\rvert^2 \geq \lvert h_{21}\rvert^2$; $\lvert h_{22}\rvert^2 \leq \lvert h_{12}\rvert^2$	$\lvert\alpha\rvert^2 \geq 1$; $\lvert\beta\rvert^2 \leq 1$
	$\lvert h_{11}\rvert^2 \leq \lvert h_{21}\rvert^2$; $\lvert h_{22}\rvert^2 \geq \lvert h_{12}\rvert^2$	$\lvert\alpha\rvert^2 \leq 1$; $\lvert\beta\rvert^2 \geq 1$

TABLE II
CLASSIFICATION OF X CHANNELS

is stronger than the corresponding direct channel. Note that as per the classification of the interference channel [9], class A, i.e., strong direct channel gain XC corresponds to the weak IC. Class B corresponds to the strong IC and classes C and D correspond to the mixed IC. The capacity region of the IC is known only in the strong interference region.

We show below that a class B channel can be converted to a class A channel and vice-versa. Further, except for an interchange in the message variables, the capacity region remains unchanged. A similar relationship exists between channels in class C and class D. In other words, the capacity regions of class B and class D can be obtained by converting them to class A and class C channels, respectively.

Consider an XC belonging to Class B, i.e., it has the following channel parameters: $|h_{11}|^2 \leq |h_{21}|^2$ and $|h_{22}|^2 \leq |h_{12}|^2$. First, we exchange the role of receiver 1 and receiver 2. This does not alter the capacity region of the XC since the output equations (1) and (2) remain the same.

Let the channel parameters be renamed as follows: $h'_{11} = h_{21}$, $h'_{12} = h_{22}$, $h'_{21} = h_{11}$, $h'_{22} = h_{12}$. This channel can now be represented in a form identical to the XC in Fig. 1 and the relationship between the channel parameters can be written as $|h'_{11}|^2 \geq |h'_{21}|^2$ and $|h'_{22}|^2 \geq |h'_{12}|^2$. Notice that this is the condition for XCs belonging to class A. The sole difference is in the interchange of the messages due to an exchange in the receivers. This shows that the capacity region remains unchanged when a class B channel is transformed to a class A channel. Using the same strategy, it can be shown that a similar relationship exists between channels in classes C and D.

Thus, we classify the XC into the two broad classes shown in Table II. The strong XC corresponds to a class of X channels where either the direct channels are stronger than the corresponding cross channels, or the cross channels are stronger than the corresponding direct channels. In the mixed XC, as the name suggests, one of the direct channels is stronger than the corresponding cross channel, whereas the other cross channel is stronger than the corresponding direct channel or vice-versa. Thus, unlike the IC, the XC can be broadly classified into just two classes. This can be intuitively explained as below. Since each transmitter communicates with both receivers, we see that in the strong cross channel gain case, each transmitter can utilize the strong cross channels to allocate appropriate rates to the cross messages. It is clear that by exchanging the receivers, this case can be mapped back to the strong direct gain XC. It is interesting to note that such a phenomenon does not happen in the IC. This is because, the cross channels always constitute interference at the receivers. Thus, we see that the strong and weak classes of the IC coalesce together in case of the XC.

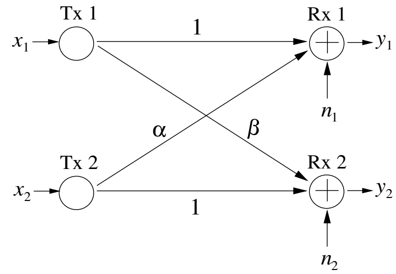

Fig. 2. Standard form XC.

A. XC in Standard Form

The XC can be written in the standard form shown in Fig. 2 [12]. The input-output equations for the standard form XC are

$$\tilde{y}_1 = \tilde{x}_1 + \alpha\,\tilde{x}_2 + \tilde{n}_1 \quad (3)$$

$$\tilde{y}_2 = \beta\,\tilde{x}_1 + \tilde{x}_2 + \tilde{n}_2, \quad (4)$$

where $\alpha = h_{12}/h_{22}$ and $\beta = h_{21}/h_{11}$. $\tilde{n}_1$ and $\tilde{n}_2$ are CSCG random variables with unit variance. The new power constraint at transmitter i is given by $\tilde{P}_i = |h_{ii}|^2 P_i$, $i = 1, 2$.

There are certain advantages to this formulation, namely, we need to deal with only two complex variables as against four for the general XC. Moreover, the relationship between the channel parameters can be characterized elegantly in the standard form XC. To illustrate this, the constraint for the strong XC in Table II can be written compactly as $|\alpha|^2 \leq 1$, $|\beta|^2 \leq 1$, or $|\alpha|^2 \geq 1$, $|\beta|^2 \geq 1$. However, in this paper, we derive all the results in terms of the actual channel parameters and do not engage the standard form XC. These results can easily be specialized to the standard form XC by substituting $h_{11} = h_{22} = 1$, $h_{12} = \alpha$ and $h_{21} = \beta$.

Using the standard form XC, the different classes of the XC can be illustrated with the help of a graph plotted in the $|\alpha|^2$–$|\beta|^2$ plane as shown in Fig. 3. Apart from strong and mixed XC regions, certain other regions can be identified. If either $\alpha = 0$ or $\beta = 0$, then the channel becomes the Z channel (see Section IV for a description of Z channels). If $\alpha\beta = 1$, then the XC is said to be degraded, and is represented by the hyperbola $|\alpha|^2|\beta|^2 = 1$. This can be easily proved as follows: When $|\alpha|^2 \geq 1$ and $|\beta|^2 \leq 1$, multiply (3) by β. It is clear that receiver 1 has a less noisier version of receiver 2's output. Thus, $\tilde{y}_2$ is a degraded version of $\tilde{y}_1$ and the sum-rate is maximized by the MAC formed by transmitters 1 and 2 to receiver 1 [11]. Similar arguments can be applied when $|\alpha|^2 \leq 1$ and $|\beta|^2 \geq 1$. In the strong XC region, in order to satisfy the condition for degradedness, we have the constraint

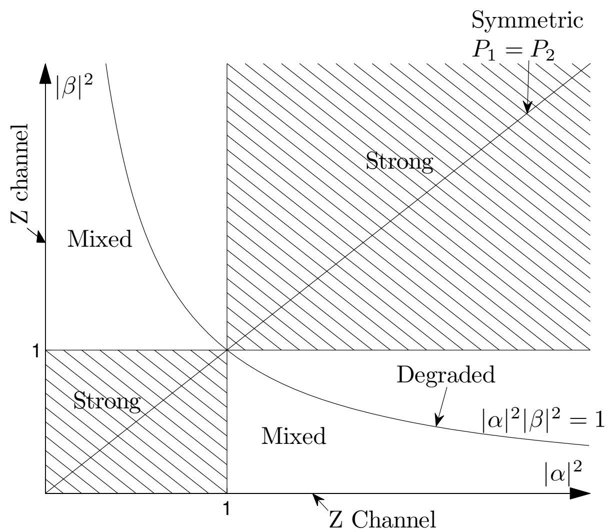

Fig. 3. Different classes of two-user XCs.

$|\alpha|^2 = |\beta|^2 = 1$. This can also be inferred from the graph since the hyperbola intersects the strong XC region only at the point $(1, 1)$. Finally, the symmetric Gaussian XC refers to the case where $P_1 = P_2$ and $|\alpha|^2 = |\beta|^2$.

Throughout the rest of the paper, strong XC refers to the X channel where the direct channels are stronger than the cross channels, i.e., $|h_{11}|^2 \geq |h_{21}|^2$ and $|h_{22}|^2 \geq |h_{12}|^2$. Similarly, mixed XC refers to X channel where $|h_{11}|^2 \geq |h_{21}|^2$ and $|h_{22}|^2 \leq |h_{12}|^2$.

IV. Z CHANNELS

In this section, we collect some results on the capacity region of the Z channel, which will be utilized to derive outer bounds on the capacity region of the XC. The Z channel is a communication system obtained from the X channel by setting the message $W_{21} = \phi$ and channel $h_{21} = 0$. Thus, there is an absence of both communication link as well as a message between transmitter 1 and receiver 2. Depending on which message and its corresponding channel are removed, there are four different Z channels associated with the X channel. They are denoted by Z(11), Z(12), Z(21) and Z(22), where Z(ij) denotes the Z channel obtained from the X channel when W_{ij} and h_{ij} are removed, $\forall\ i, j \in \{1, 2\}$. The Z(21) channel is shown in Fig. 4.

In the following, we state some capacity results for the Z(21) channel. These can be easily extrapolated to other Z channels by first writing them in a form similar to the Z(21) and substituting for the corresponding variables. For the Z channel, different types of degradation can be defined as in [14]. We focus on type I and type II degradations below [14].

Definition 1. We define a ZC to be degraded of type I if $x_2 \to (x_1, y_2) \to y_1$ form a Markov chain. It is shown in [14] that this class of degraded ZCs is equivalent to the condition that $|h_{12}|^2 < |h_{22}|^2$ and the symbol x_2 received at receiver 1 is a degraded version of x_2 received at receiver 2.

Definition 2. We define a ZC to be degraded of type II if $x_2 \to (x_1, y_1) \to y_2$ form a Markov chain. It is shown in [14] that this class of degraded ZCs is equivalent to the condition that $|h_{12}|^2 \geq |h_{22}|^2$ and the symbol x_2 received at receiver 2 is a degraded version of x_2 received at receiver 1.

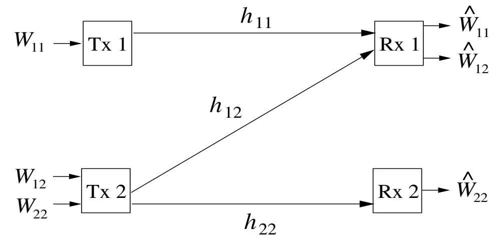

Fig. 4. Z channel

We have the following outer bounds on the capacity region of type I and type II degraded ZCs. An outer bound to the capacity region of the type I degraded Gaussian ZC is determined in [13], which we state below.

Theorem 1 (Liu and Ulukus). For the degraded Gaussian ZC of type I, with power constraints P_1 and P_2, the achievable rate triplet (R_{11}, R_{12}, R_{22}) has to satisfy

$$R_{11} \leq \log(1 + |h_{11}|^2 P_1) \quad (5)$$

$$R_{12} \leq \log\left(1 + \frac{|h_{12}|^2 p_{12}}{1 + |h_{12}|^2 p_{22}}\right) \quad (6)$$

$$R_{22} \leq \log(1 + |h_{22}|^2 p_{22}) \quad (7)$$

$$R_{11} + R_{12} \leq \log\left(1 + \frac{|h_{11}|^2 P_1 + |h_{12}|^2 p_{12}}{1 + |h_{12}|^2 p_{22}}\right), \quad (8)$$

for some $0 \leq p_{12} \leq P_2$ and $p_{22} = P_2 - p_{12}$.

Proof: See [13, Theorem 2 and Section V-B]. Interestingly, in [13, Theorem 1], the authors present an achievable scheme which is able to achieve the bounds in (6)-(8). Thus, the bounds in (6)-(8) are in fact tight. □

In [14], an outer bound to the capacity region of the type II degraded Gaussian ZC is determined, which we state below.

Theorem 2 (Chong et al.). For the degraded Gaussian ZC of type II, with power constraints P_1 and P_2, the achievable rate triplet (R_{11}, R_{12}, R_{22}) has to satisfy

$$R_{11} \leq \log(1 + |h_{11}|^2 P_1) \quad (9)$$

$$R_{12} \leq \log(1 + |h_{12}|^2 p_{12}) \quad (10)$$

$$R_{22} \leq \log\left(1 + \frac{|h_{22}|^2 p_{22}}{1 + |h_{22}|^2 p_{12}}\right) \quad (11)$$

$$R_{11} + R_{12} + R_{22} \leq \log(1 + |h_{11}|^2 P_1 + |h_{12}|^2 P_2), \quad (12)$$

for some $0 \leq p_{12} \leq P_2$ and $p_{22} = P_2 - p_{12}$.

Proof: See [14, Theorem 7]. The authors also give an achievable region in [14, Corollary 2]. □

V. OUTER BOUNDS ON THE CAPACITY REGION OF XC

We make use of Theorem 1 and Theorem 2 to derive outer bounds on the capacity region of the XC.

Remark 1. In [15], an outer bound is obtained for weak Gaussian IC. The outer bound relies on the fact that removing one of the interfering links enlarges the capacity region of the IC. The capacity region of the IC is contained within the intersection of the capacity regions of the two one-sided Gaussian ICs. Although the capacity region of the one-sided Gaussian IC is unknown, Kramer makes use of an outer bound due to Sato [16] to derive an outer bound for weak Gaussian IC.

This approach cannot be directly applied to the XC since, unlike the IC, the cross channels also carry messages apart from interference from unintended signals. However, interestingly, an analogous result can in fact be derived for the XC. It is based on the idea that when an XC is converted to a ZC by removing one of the communication links and the corresponding message, the rate region of the remaining messages are enlarged. Since the capacity region of the ZC is not known, we use the outer bounds described in Theorem 1 and Theorem 2 instead. We prove this result in the following theorem.

Theorem 3. Consider a XC in which the link h_{21} and the message W_{21} are removed to obtain the Z(21) channel. Then the rate region of the XC with respect to the rate triplet (R_{11}, R_{12}, R_{22}) is contained within the outer bounds to the capacity region of the Z(21) channel, described in Theorem 1 and Theorem 2, with a power allocation of p_{11} at transmitter 1 and P_2 at transmitter 2, where $0 \leq p_{11} \leq P_1$.

Proof: See the Appendix. □

Remark 2. Let $\mathcal{R}_{21}$ be a subspace in $\mathbb{R}_+^4$ with the rate triplet (R_{11}, R_{12}, R_{22}) bounded by Theorem 3 and rate variable R_{21} unbounded. Applying Theorem 3, it is clear from the above definition that the set $\mathcal{R}_{21}$ defines an outer bound to the capacity region of XC. Although Theorem 3 has been proved with respect to the Z(21) channel, similar results apply to the other 3 combinations of rate triplets of the XC. For example, the rate region of the XC with respect to the rate triplet (R_{11}, R_{21}, R_{22}) is contained within outer bounds to the capacity region of the Z(12) channel given in Theorem 1 and Theorem 2 with a power allocation of P_1 at transmitter 1 and $0 \leq p_{22} \leq P_2$ at transmitter 2, where Theorems 1 and 2 have been appropriately modified for the $Z(12)$ channel. $\mathcal{R}_{12}$ denotes a subspace in $\mathbb{R}_+^4$ with (R_{11}, R_{21}, R_{22}) bounded as above and R_{12} left unbounded. Regions $\mathcal{R}_{11}$ and $\mathcal{R}_{22}$ can be defined similarly.

Let $\mathcal{R}$ denote the capacity region of the XC. Then, we have the following theorem.

Theorem 4. The capacity region of the Gaussian XC is contained within the set $\mathcal{R}_I$, i.e., $\mathcal{R} \subset \mathcal{R}_I$, where

$$\mathcal{R}_I = \mathcal{R}_{11} \cap \mathcal{R}_{12} \cap \mathcal{R}_{21} \cap \mathcal{R}_{22}. \tag{13}$$

Proof: Using Remark 2 and Theorem 3, each of the sets $\mathcal{R}_{ij}$ defines an outer bound to the capacity region of the XC, $\forall i,\ j \in 1, 2$. Therefore, the capacity region of the Gaussian XC is inside the intersection of the outer bounds to the capacity regions of XC given in (13). □

VI. Strong Gaussian X Channel

In this section, we derive the outer bound for the capacity region of the strong Gaussian XC and use this to derive a new sum-rate outer bound.

A. Outer Bound on the Capacity Region

Theorem 5. The capacity region of the strong Gaussian XC is contained within the set of rate vectors $(R_{11}, R_{12}, R_{21}, R_{22})$ satisfying

$$\begin{aligned}
R_{11} &\leq \psi_1 = \log(1 + |h_{11}|^2 p_{11}) \\
R_{12} &\leq \psi_2 = \log\left(1 + \frac{|h_{12}|^2 p_{12}}{1 + |h_{12}|^2 p_{22}}\right) \\
R_{21} &\leq \psi_3 = \log\left(1 + \frac{|h_{21}|^2 p_{21}}{1 + |h_{21}|^2 p_{11}}\right) \\
R_{22} &\leq \psi_4 = \log(1 + |h_{22}|^2 p_{22}) \\
R_{11} + R_{12} &\leq \psi_5 = \log\left(1 + \frac{|h_{11}|^2 p_{11} + |h_{12}|^2 p_{12}}{1 + |h_{12}|^2 p_{22}}\right) \\
R_{21} + R_{22} &\leq \psi_6 = \log\left(1 + \frac{|h_{21}|^2 p_{21} + |h_{22}|^2 p_{22}}{1 + |h_{21}|^2 p_{11}}\right) \\
R_{11} + R_{12} + R_{21} &\leq \psi_7 = \log(1 + |h_{11}|^2 P_1 + |h_{12}|^2 p_{12}) \\
R_{12} + R_{21} + R_{22} &\leq \psi_8 = \log(1 + |h_{21}|^2 p_{21} + |h_{22}|^2 P_2),
\end{aligned} \tag{14}$$

for some $0 \leq p_{11} \leq P_1$, $0 \leq p_{12} \leq P_2$ with $p_{21} = P_1 - p_{11}$, $p_{22} = P_2 - p_{12}$.

Proof: The rate equations in (14) is the representation of the set $\mathcal{R}_I$ given in Theorem 4 for the strong Gaussian XC. The rate equations can be obtained by applying Theorem 3 to the four Z channels associated with the strong Gaussian XC and removing redundant equations. □

Let $R_1 = R_{11} + R_{12}$ denote the rate at receiver 1 and let $R_2 = R_{21} + R_{22}$ denote the rate at receiver 2. The outer bounds in (14) can be converted to a set of outer bounds on R_1, R_2 and $R_1 + R_2$ by using Fourier-Motzkin elimination and removing the redundant equations. The result is as follows.

Theorem 6. The capacity region of the strong Gaussian XC is contained within the set of rate pairs (R_1, R_2) satisfying

$$R_1 \leq \psi_5 \tag{15}$$

$$R_2 \leq \psi_6 \tag{16}$$

$$R_1 + R_2 \leq \psi_1 + \psi_8 \tag{17}$$

$$R_1 + R_2 \leq \psi_4 + \psi_7, \tag{18}$$

for some $0 \leq p_{11} \leq P_1$, $0 \leq p_{12} \leq P_2$ with $p_{21} = P_1 - p_{11}$, $p_{22} = P_2 - p_{12}$.

B. New Sum-Rate Outer Bound

Applying Theorem 6, the sum-rate of strong Gaussian XC is bounded by three different rate-inequalities, namely (17), (18), and a combination of (15), (16), Thus, three sum-rate outer bounds can be derived by maximizing each of the above rate-inequalities over (p_{11}, p_{12}). However, it can be shown that the latter bound is tighter than the former two bounds. Hence, we describe only the last bound below.

The sum-rate outer bound is given by $\max_{p_{11},\ p_{12}}(\psi_5+\psi_6)$. It appears as though we need to perform joint optimization over (p_{11}, p_{12}) of a non-convex objective function. Fortunately, this can be rewritten so that the maximizations can be carried out independently. We have

$$\max_{p_{11}} \log\left(\frac{|h_{11}|^2 p_{11} + |h_{12}|^2 P_2 + 1}{1 + |h_{21}|^2 p_{11}}\right) + \max_{p_{22}} \log\left(\frac{1 + |h_{21}|^2 P_1 + |h_{22}|^2 p_{22}}{1 + |h_{12}|^2 p_{22}}\right). \quad (19)$$

Although we have decoupled the optimization variables in (19), each of the individual maximizations still represents a maximization over a non-convex objective function. We make use of the following lemma to solve (19).

Lemma 1. Define the function $f(x) = \log\left(\frac{1+cx}{1+dx}\right)$, where $x, c, d \in \mathbb{R}_+$. Then, the solution of the maximization problem

$$x^* = \arg\max_{0 \le x \le P} f(x) = \arg\max_{0 \le x \le P} \log\left(\frac{1+cx}{1+dx}\right)$$

is given by

$$x^* = \begin{cases} P & \text{if } c > d, \\ 0 & \text{if } c < d. \end{cases}$$

and when $c = d$, any value of $x^* \in [0, P]$ can be chosen.

Proof: The lemma can be easily proved by observing the monotonicity of the function $f(x)$. The function $f(x)$ is continuous and differentiable $\forall x \in [0, P]$. Observe that the derivative $f'(x)$ is positive if $c > d$ and negative if $c < d$. This implies that if $c > d$, $f(x)$ is a *strictly increasing* function on the interval $[0, P]$ and the maximum of $f(x)$ is achieved when $x^* = P$. Conversely, when $c < d$, $f(x)$ is a *strictly decreasing* function and is maximized when $x^* = 0$. When $c = d$, $f(x) = 0$, $\forall x \in [0, P]$ and any value of $x^* \in [0, P]$ can be chosen. □

Writing the first maximization in (19) in the form of the above lemma, it is clear that, when $|h_{11}|^2 \ge |h_{21}|^2(1 + |h_{12}|^2 P_2)$, $p_{11} = P_1$ solves the first maximization problem. Similarly, if $|h_{22}|^2 \ge |h_{12}|^2(1 + |h_{21}|^2 P_1)$, then $p_{22} = P_2$. Substituting this in (19) and rearranging, we get

$$R \le \log\left(1 + \frac{|h_{11}|^2 P_1}{1 + |h_{12}|^2 P_2}\right) + \log\left(1 + \frac{|h_{22}|^2 P_2}{1 + |h_{21}|^2 P_1}\right).$$

Similarly, sum-rate bounds can be calculated for the other three sub-regions. These results are summarized in Table III.

C. Sum-Rate Capacity

We show that the sum-rate outer bound given in Table III is tight in three out of the four sub-regions.

Theorem 7. The sum-rate outer bound for the strong Gaussian XC given in Table III is achievable in regions I, II and III, and it represents the sum-rate capacity in those regions.

Proof: The sum-rate outer bound for region I given in Table III can be achieved with a simple scheme of transmitting only on the direct channels, i.e., W_{11} and W_{22} are transmitted with power P_1 and P_2, respectively, while the cross messages $W_{12} = W_{21} = \phi$. The cross channels interfere with the decoding of the direct messages. Such a scheme clearly achieves the sum-rate bound for region I. The bound for region II can be achieved with the MAC transmission from transmitters 1 and 2 to receiver 1, while messages $W_{21} = W_{22} = \phi$. Similarly, the outer bound for region III can be achieved with the MAC transmission from transmitters 1 and 2 to receiver 2, while messages $W_{11} = W_{12} = \phi$. Thus, we have shown that the sum-rate outer bounds for regions I, II and III are achievable. □

Observe that in regions II and III, the XC is operated as a MAC, while in region I, the XC is operated as an IC. The outer bound to the capacity region for a strong Gaussian XC is illustrated in Fig. 5 for all the four regions. It is clear from the rate equations in Theorem 6 that there are at most five corner points. We calculate the corner points below. Point A is obtained by maximizing (15) over all (p_{11}, p_{12}). It is obvious that $p_{11} = P_1$, $p_{12} = P_2$ solves this problem and the corner point A is given by

$$R_1 \le \log(1 + |h_{11}|^2 P_1 + |h_{12}|^2 P_2),$$

which is achievable by the MAC formed by transmitters 1 and 2 to receiver 1. Similarly, corner point D is given by

$$R_2 \le \log(1 + |h_{21}|^2 P_1 + |h_{22}|^2 P_2),$$

and is achieved by the MAC formed by transmitters 1 and 2 to receiver 2.

In Fig. 5a, the line BC represents the sum-rate bound for region I given in Table III and point F is the rate point achieved by transmitting only on the direct channels and treating interference as noise. Theorem 7 ensures us that point F lies on BC, i.e., sum-rate capacity is achieved at point F. The line AF can be achieved using a time-sharing strategy between the points A and F. Similarly, line DF is also achievable. Thus, any point within the region AEDFA is achievable and the shaded region denotes an achievable region for region I. Likewise, the outer bounds for the other regions are illustrated in Fig. 5b–Fig. 5d.

D. Discussion on the Implications of Region I

In region I, messages are transmitted only on the direct channels and interference from the cross channels are treated as noise. This is akin to the noisy-interference or the low-interference sum-rate capacity result of the IC. In this region, sum-rate capacity of the IC is achieved by treating interference as noise [8]–[10]. In [7], the authors showed that the results carryover to the XC, i.e., the XC can be operated as an IC and interference treated as noise to achieve the sum-rate capacity. The low-interference sum-rate capacity region for the IC/XC given in [7]–[10] is

$$\frac{|h_{21}|}{|h_{11}|}(1 + |h_{12}|^2 P_2) + \frac{|h_{12}|}{|h_{22}|}(1 + |h_{21}|^2 P_1) \le 1. \quad (20)$$

On the other hand, the channel constraints for region I can be rewritten as

$$\frac{|h_{21}|^2}{|h_{11}|^2}(1 + |h_{12}|^2 P_2) \le 1 \ ; \ \frac{|h_{12}|^2}{|h_{22}|^2}(1 + |h_{21}|^2 P_1) \le 1. \quad (21)$$

Region	Channel Constraints		Sum-Rate Outer Bound for Strong Gaussian XC
I	$\lvert h_{11}\rvert^2 \geq \lvert h_{21}\rvert^2(1+\lvert h_{12}\rvert^2 P_2)$	$\lvert h_{22}\rvert^2 \geq \lvert h_{12}\rvert^2(1+\lvert h_{21}\rvert^2 P_1)$	$\log\left(1+\frac{\lvert h_{11}\rvert^2 P_1}{1+\lvert h_{12}\rvert^2 P_2}\right)+\log\left(1+\frac{\lvert h_{22}\rvert^2 P_2}{1+\lvert h_{21}\rvert^2 P_1}\right)$
II	$\lvert h_{11}\rvert^2 \geq \lvert h_{21}\rvert^2(1+\lvert h_{12}\rvert^2 P_2)$	$\lvert h_{22}\rvert^2 \leq \lvert h_{12}\rvert^2(1+\lvert h_{21}\rvert^2 P_1)$	$\log(1+\lvert h_{11}\rvert^2 P_1+\lvert h_{12}\rvert^2 P_2)$
III	$\lvert h_{11}\rvert^2 \leq \lvert h_{21}\rvert^2(1+\lvert h_{12}\rvert^2 P_2)$	$\lvert h_{22}\rvert^2 \geq \lvert h_{12}\rvert^2(1+\lvert h_{21}\rvert^2 P_1)$	$\log(1+\lvert h_{21}\rvert^2 P_1+\lvert h_{22}\rvert^2 P_2)$
IV	$\lvert h_{11}\rvert^2 \leq \lvert h_{21}\rvert^2(1+\lvert h_{12}\rvert^2 P_2)$	$\lvert h_{22}\rvert^2 \leq \lvert h_{12}\rvert^2(1+\lvert h_{21}\rvert^2 P_1)$	$\log(1+\lvert h_{12}\rvert^2 P_2)+\log(1+\lvert h_{21}\rvert^2 P_1)$

TABLE III
SUM-RATE BOUND FOR STRONG GAUSSIAN X CHANNELS

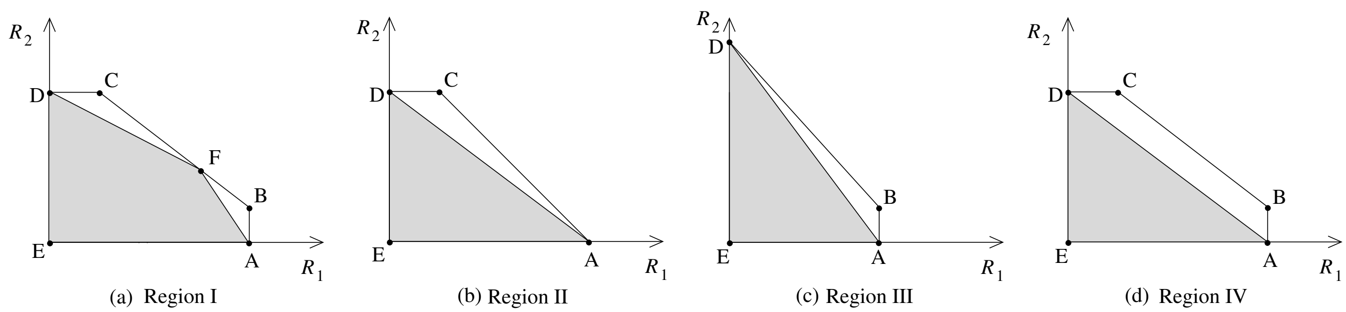

Fig. 5. Illustration of the outer bound to the capacity region of strong Gaussian XC for all the four regions defined in Table III.

Curiously, it is not difficult to see that the region defined by channel constraints in (21) is a larger region than (20) and includes (20) as a subset. Since the XC is operated as an IC in region I, and both of them share the same physical channel, this begs the question whether region I defined by (21) carries over to the IC. Note that (20) is only a sufficient condition for the IC to be in the low-interference regime, where treating interference as noise achieves sum-rate capacity. Thus, there might exist channels which do not satisfy (20), but belong to the low-interference regime [10].

Although we suspect that (21) is indeed the new sufficient condition for an IC to be in the low-interference region, to conclusively settle the argument, we need to come up with tight sum-rate outer bounds for the IC to be in the low-interference region. We do not pursue this here as this will take us away from the XC, which is the focus of this paper.

E. Comparison with Other Sum-Rate Outer Bounds

The only known sum-rate outer bounds for the XC are the ETW bounds in [7, Theorem 5.3]. With some effort it is not difficult to show that the sum-rate outer bound for strong Gaussian XCs in Table III outperforms the ETW bounds in all regions of the strong XC sum-rate capacity region.

VII. MIXED GAUSSIAN X CHANNEL

Similar to the previous section, we derive outer bounds on the capacity region of the mixed Gaussian XC and use this to derive two sum-rate outer bounds.

A. Outer Bound on the Capacity Region

Theorem 8. The capacity region of the mixed Gaussian XC is contained within the set of rate vectors $(R_{11}, R_{12}, R_{21}, R_{22})$ satisfying

$$\begin{aligned}
R_{11} &\leq \varphi_1 = \log(1+\lvert h_{11}\rvert^2 p_{11})\\
R_{12} &\leq \varphi_2 = \log(1+\lvert h_{12}\rvert^2 p_{12})\\
R_{21} &\leq \varphi_3 = \log\left(1+\frac{\lvert h_{21}\rvert^2 p_{21}}{1+\lvert h_{21}\rvert^2 p_{11}}\right)\\
R_{22} &\leq \varphi_4 = \log\left(1+\frac{\lvert h_{22}\rvert^2 p_{22}}{1+\lvert h_{22}\rvert^2 p_{12}}\right)\\
R_{21}+R_{22} &\leq \varphi_5 = \log\left(1+\frac{\lvert h_{21}\rvert^2 p_{21}+\lvert h_{22}\rvert^2 p_{22}}{1+\lvert h_{21}\rvert^2 p_{11}}\right)\\
R_{21}+R_{22} &\leq \varphi_6 = \log\left(1+\frac{\lvert h_{21}\rvert^2 p_{21}+\lvert h_{22}\rvert^2 p_{22}}{1+\lvert h_{22}\rvert^2 p_{12}}\right)\\
R_{11}+R_{12}+R_{21} &\leq \varphi_7 = \log(1+\lvert h_{11}\rvert^2 P_1+\lvert h_{12}\rvert^2 p_{12})\\
R_{11}+R_{12}+R_{22} &\leq \varphi_8 = \log(1+\lvert h_{11}\rvert^2 p_{11}+\lvert h_{12}\rvert^2 P_2).
\end{aligned} \tag{22}$$

for some $0 \leq p_{11} \leq P_1$, $0 \leq p_{12} \leq P_2$ with $p_{21} = P_1 - p_{11}$, $p_{22} = P_2 - p_{12}$.

Proof: The rate equations in (22) is the representation of the set $\mathcal{R}_I$ given in Theorem 4 for the mixed Gaussian XC. The rate equations can be obtained by applying Theorem 3 to the four Z channels associated with the mixed Gaussian XC and removing redundant equations. □

The outer bounds in (22) can be converted to a set of outer bounds on R_1, R_2, R_1+R_2 and $2R_1+R_2$ by using Fourier-Motzkin elimination. The result is as follows.

Theorem 9. The capacity region of the mixed Gaussian XC is contained within the set of rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq \varphi_1+\varphi_2 && (23)\\ R_2 &\leq \varphi_5 && (24)\\ R_2 &\leq \varphi_6 && (25)\\ R_2 &\leq \varphi_3+\varphi_4 && (26)\\ R_1+R_2 &\leq \varphi_4+\varphi_7 && (27)\\ R_1+R_2 &\leq \varphi_3+\varphi_8 && (28)\\ 2R_1+R_2 &\leq \varphi_7+\varphi_8, && (29)\end{aligned}$$

for some $0 \leq p_{11} \leq P_1$, $0 \leq p_{12} \leq P_2$ with $p_{21} = P_1 - p_{11}$, $p_{22} = P_2 - p_{12}$.

B. New Sum-Rate Outer Bounds

As in the case of the strong Gaussian XC, using Theorem 9, two sum-rate outer bounds on the mixed Gaussian XC can be derived. We describe them one by one below. The first bound is given by $\max_{p_{11},\ p_{12}}(\varphi_4+\varphi_7)$ and can be written as

$$R \leq \max_{p_{11},\ p_{12}} \log\left(\frac{(1+|h_{11}|^2P_1+|h_{12}|^2p_{12})(1+|h_{22}|^2P_2)}{1+|h_{22}|^2p_{12}}\right).$$

Applying Lemma 1 to the above expression, we conclude that if $|h_{12}|^2 \geq |h_{22}|^2(1+|h_{11}|^2P_1)$ then $p_{12} = P_2$ and we have

$$R \leq \log(1+|h_{11}|^2P_1+|h_{12}|^2P_2). \quad (30)$$

On the other hand, if $|h_{12}|^2 \leq |h_{22}|^2(1+|h_{11}|^2P_1)$, then

$$R \leq \log(1+|h_{11}|^2P_1)+\log(1+|h_{22}|^2P_2). \quad (31)$$

The second outer bound can be written as $\max_{p_{11},\ p_{12}}(\varphi_3+\varphi_8)$. Using a similar analysis as in the first bound, we conclude that if $|h_{11}|^2 \geq |h_{21}|^2(1+|h_{12}|^2P_2)$, the sum-rate is bounded by (30). If this condition is not met, then the sum-rate is bounded as

$$R \leq \log(1+|h_{12}|^2P_2)+\log(1+|h_{21}|^2P_1). \quad (32)$$

The above results are summarized in Table IV. Finally, $2R_1+R_2$ in (29) is bounded by $\max_{p_{11},\ p_{12}}(\varphi_7+\varphi_8)$ and is given by

$$2R_1+R_2 \leq 2\log(1+|h_{11}|^2P_1+|h_{12}|^2P_2), \quad (33)$$

which is achievable by the MAC formed by transmitters 1 and 2 to receiver 1 over two channel uses.

C. Sum-Rate Capacity

From the discussions in the previous subsection, we have the following theorem.

Theorem 10. The sum-rate outer bound for the mixed Gaussian XC given in Table IV is achievable in regions I and II, and it represents the sum-rate capacity in those regions.

Proof: The sum-rate bound for regions I and II in Table IV is achieved by the MAC formed by transmitters 1 and 2 to receiver 1. □

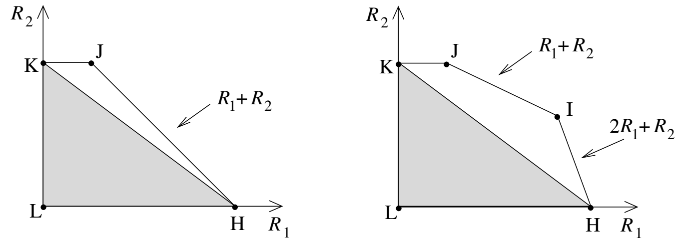

Fig. 6. Illustration of the outer bound to the capacity region of mixed Gaussian XC for all the three regions defined in Table IV

The above sum-rate capacity result was first obtained in [11, Theorem 4] and a rather long three part proof was given to prove each of the regions in the above theorem. However, we have shown that the regions given in Theorem 10 are obtained as a natural consequence of the outer bound for the capacity region of the mixed Gaussian XC given in Theorem 9.

Note that unlike in case of the strong Gaussian XC, the channel constraints given in regions I and II in Table IV are independent of each other. This means that when either of the regions are true, as per Theorem 10, the sum-rate capacity is achieved. It is clear that the remaining region can be characterized by the intersection of the channel constraints in region III in Table IV, and in this case, the sum-rate is bounded by the minimum of the rate inequalities in (31) and (32).

The outer bound to the capacity region is illustrated in Fig. 6 for all the three regions. In all the graphs, point K is obtained by maximizing (24)-(26) over all (p_{11}, p_{12}) and taking their minimum. It is given by

$$R_2 \leq \log(1+|h_{21}|^2P_1+|h_{22}|^2P_2). \quad (34)$$

Similarly, point H is obtained from (33) which gives a tighter bound on rate R_1 than that obtained by maximizing (23) over all (p_{11}, p_{12}) and is given by

$$R_1 \leq \log(1+|h_{11}|^2P_1+|h_{12}|^2P_2). \quad (35)$$

Fig. 6a represents the outer bound to the capacity region for regions I and II. The shaded region represents an achievable region for regions I and II. This is because, points H and K are achievable by the MAC at receiver 1 and receiver 2, respectively. The line KH is achievable by time-sharing between these two strategies. Fig. 6b represents the outer bound to the capacity region for region III and the shaded region represents an achievable region.

D. Comparison with Other Known Sum-Rate Outer Bounds

We compare the outer bounds developed in the previous subsection with the ETW bounds in [7, Theorem 5.3]. With some effort it is not difficult to show that the sum-rate outer bounds in the previous subsection outperform the ETW bounds in regions I and II. In region III, further analysis is needed to ascertain the comparative tightness of the bounds.

Region	Channel Constraints		Sum-Rate Outer Bound for Mixed Gaussian XC
I	$\lvert h_{12}\rvert^2 \geq \lvert h_{22}\rvert^2(1+\lvert h_{11}\rvert^2 P_1)$	-	$\log(1+\lvert h_{11}\rvert^2 P_1+\lvert h_{12}\rvert^2 P_2)$
II	-	$\lvert h_{11}\rvert^2 \geq \lvert h_{21}\rvert^2(1+\lvert h_{12}\rvert^2 P_2)$	$\log(1+\lvert h_{11}\rvert^2 P_1+\lvert h_{12}\rvert^2 P_2)$
III	$\lvert h_{12}\rvert^2 \leq \lvert h_{22}\rvert^2(1+\lvert h_{11}\rvert^2 P_1)$	$\lvert h_{11}\rvert^2 \leq \lvert h_{21}\rvert^2(1+\lvert h_{12}\rvert^2 P_2)$	$\min\big(\log(1+\lvert h_{11}\rvert^2 P_1)+\log(1+\lvert h_{22}\rvert^2 P_2), \log(1+\lvert h_{12}\rvert^2 P_2)+\log(1+\lvert h_{21}\rvert^2 P_1)\big)$

TABLE IV
SUM-RATE OUTER BOUND FOR MIXED GAUSSIAN X CHANNELS

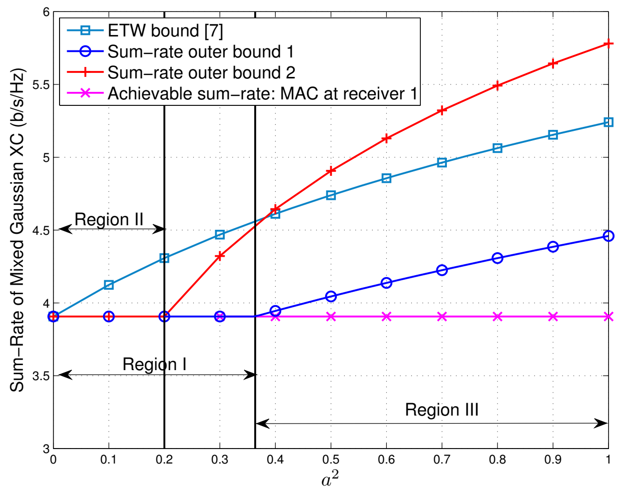

Fig. 7. Comparison of sum-rate outer bounds for mixed Gaussian X channel.

In Fig. 7, we compare the sum-rate outer bounds in the previous subsection with the ETW bounds in [7], where we have plotted the minimum of the two bounds in [7, Theorem 5.3]. We assume the following: $h_{11}=1$, $h_{12}=2$, $h_{21}=h_{22}=a$, $P_1=10$dB, $P_2=0$dB. We plot the performance of the bounds when a^2 is varied from 0 to 1.

Also plotted are the three regions defined in Table IV. The channel constraint for region I is given by $a^2 \leq 4/(1+P_1) = 0.363$ and that for region II is $a^2 \leq 1/(1+4P_2) = 0.2$. Thus, for this particular channel configuration and power levels, we see that region II is contained within region I and the first sum-rate outer bound outperforms the ETW bound in all the three regions.

VIII. CONCLUSIONS

We investigated the capacity region of the Gaussian X channel (XC). We first classified the XC into two classes, the strong XC and the mixed XC. We derived bounds on the capacity region for each of the two classes. We used the idea that when one of the messages is eliminated from the XC, the rate region of the remaining three messages are enlarged. We made use of the Z channel to bound the rate region of the remaining messages and showed that it defines a subspace in $\mathbb{R}_+^4$ and forms an outer bound to the capacity region of the XC. Thus, the outer bound to the capacity region of the XC was obtained as the intersection of the outer bounds to the four combinations of the rate triplets of the XC. Using these outer bounds, we derived new sum-rate outer bounds for both strong and mixed Gaussian XCs and compared them with those existing in literature. We showed that the sum-rate outer bound for strong XC gave the sum-rate capacity in three out of the four sub-regions of the strong XC capacity region. In case of mixed XC, we recovered the recent sum-rate capacity results in [11] and gave a simple alternate proof of the same.

APPENDIX
PROOF OF THEOREM 3

Consider the XC with the message W_{21} given at both the receivers as side information. This is equivalent to removing link h_{21} and the message W_{21} from the XC. Thus, we obtain the Z(21) channel.

Let $|h_{22}|^2 > |h_{12}|^2$ which is the condition for the type I degradation of the resulting Z(21) channel. Applying Theorem 1 to this channel, clearly (5)–(8) represents the rate region of the remaining messages W_{11}, W_{12} and W_{22}. It remains to prove the existence of $p_{11} \in [0, P_1]$ such that the rate equations (5)–(8) continue to hold true when P_1 is replaced with p_{11}. Since (6) and (7) do not contain P_1, it suffices to show that the power allocation P_1 in (5) and (8) can be replaced with $0 \leq p_{11} \leq P_1$.

To this end, let $\mathbf{y}_i^n$ denote the vector of received symbols of length n at receiver i. Let $\mathbf{x}_i^n$ denote the n length vector of transmitted symbols at transmitter i. Along with message W_{21}, let $\mathbf{x}_2^n$ also be made available at receiver 1. Using Fano's inequality, we can bound R_{11} as follows

$$\begin{aligned} nR_{11} &\leq I(W_{11}\,;\mathbf{y}_1^n,\, W_{21},\, \mathbf{x}_2^n)+n\epsilon_{1n} \\ &\leq I(W_{11}\,;\mathbf{y}_1^n \,|\, W_{21},\, \mathbf{x}_2^n)+n\epsilon_{1n} \\ &\leq h(\mathbf{y}_1^n \,|\, W_{21},\, \mathbf{x}_2^n)-h(\mathbf{n}_1^n)+n\epsilon_{1n}, \end{aligned} \quad (36)$$

where the last but one inequality follows since W_{21}, $\mathbf{x}_2^n$ are independent of W_{11} and $\epsilon_{1n} \to 0$ as $n \to \infty$. Next we bound the term $h(\mathbf{y}_1^n \,|\, W_{21},\, \mathbf{x}_2^n)$. Consider the following set of inequalities

$$n\log(\pi e) \overset{(a)}{=} h(\mathbf{n}_1^n) = h(\mathbf{y}_1^n \,|\, \mathbf{x}_1^n,\, \mathbf{x}_2^n) \quad (37)$$

$$\begin{aligned} &\overset{(b)}{\leq} h(\mathbf{y}_1^n \,|\, W_{21},\, \mathbf{x}_2^n) \\ &\overset{(c)}{\leq} h(\mathbf{y}_1^n \,|\, \mathbf{x}_2^n) \\ &\overset{(d)}{\leq} n\log(\pi e(1+|h_{11}|^2 P_1)), \end{aligned} \quad (38)$$

where in steps (a) and (d), we use the fact that the circularly symmetric complex Gaussian distribution maximizes the differential entropy for a given covariance constraint, steps (b) and (c) follow since removing conditioning cannot reduce differential entropy. Observe that the term $h(\mathbf{y}_1^n \,|\, W_{21},\, \mathbf{x}_2^n)$ is upper bounded by (38) and lower bounded by (37). Hence, we conclude that there exists $p_{11} \in [0, P_1]$, such that

$$h(\mathbf{y}_1^n \,|\, W_{21},\, \mathbf{x}_2^n) \;=\; n\log(\pi e(1+|h_{11}|^2 p_{11})). \quad (39)$$

Using (39) and (37) in (36), we get

$$R_{11} \;\le\; \log(1+|h_{11}|^2 p_{11}) + \epsilon_{1n}, \quad (40)$$

and as $n \to \infty$, $\epsilon_{1n} \to 0$ and we get the desired bound.

We next develop a bound for $R_{11} + R_{12}$ below.

$$\begin{aligned} n(R_{11}+R_{12}) &\le I(W_{11},\, W_{12}\,;\, \mathbf{y}_1^n,\, W_{21}) + n\epsilon_{2n} \\ &\le I(W_{11},\, W_{12}\,;\, \mathbf{y}_1^n \,|\, W_{21}) + n\epsilon_{2n} \\ &\le h(\mathbf{y}_1^n \,|\, W_{21}) - h(\mathbf{y}_1^n \,|\, \mathbf{x}_1^n, W_{12}) + n\epsilon_{2n}. \quad (41) \end{aligned}$$

Using the same strategy as (38), we can bound each of the conditional entropy terms in (41) as follows:

$$\begin{aligned} n\log(\pi e) &= h(\mathbf{y}_1^n \,|\, \mathbf{x}_1^n,\, \mathbf{x}_2^n) \\ &\le h(\mathbf{y}_1^n \,|\, W_{21}) \\ &\le h(\mathbf{y}_1^n) \\ &\le n\log(\pi e(1+|h_{11}|^2 P_1 + |h_{12}|^2 P_2)). \end{aligned}$$

Thus, there exists $p_{11} \in [0, P_1]$, such that

$$h(\mathbf{y}_1^n \,|\, W_{21}) \;=\; \log(\pi e(1+|h_{11}|^2 p_{11} + |h_{12}|^2 P_2)). \quad (42)$$

Similarly, we bound the term $h(\mathbf{y}_1^n \,|\, \mathbf{x}_1^n,\, W_{12})$ as given below.

$$\begin{aligned} n\log(\pi e) &= h(\mathbf{y}_1^n \,|\, \mathbf{x}_1^n,\, \mathbf{x}_2^n) \\ &\le h(\mathbf{y}_1^n \,|\, \mathbf{x}_1^n,\, W_{12}) \\ &\le h(\mathbf{y}_1^n \,|\, \mathbf{x}_1^n) \\ &\le n\log(\pi e(1+|h_{12}|^2 P_2)). \end{aligned}$$

From the above set of inequalities, we conclude that there exists $p_{22} \in [0, P_2]$, such that

$$h(\mathbf{y}_1^n \,|\, \mathbf{x}_1^n,\, W_{12}) \;=\; \log(\pi e(1+|h_{12}|^2 p_{22})). \quad (43)$$

Substituting (42) and (43) in (41) and simplifying, we get

$$R_{11} + R_{12} \;\le\; \log\left(1 + \frac{|h_{11}|^2 p_{11} + |h_{12}|^2 p_{12}}{1+|h_{12}|^2 p_{22}}\right) + n\epsilon_{2n},$$

where we have defined $p_{12} = P_2 - p_{22}$ and as $n \to \infty$, $\epsilon_{2n} \to 0$ and we get the desired bound.

Next, let $|h_{22}|^2 \le |h_{12}|^2$ which is the condition for the type II degradation of the resulting Z(21) channel. Applying Theorem 2 to this channel, clearly (9)–(12) represents the rate region of the remaining messages W_{11}, W_{12} and W_{22}. Since (9) is the same as (5), and (6), (7) do not contain P_1, it suffices to show that the power allocation P_1 in (12) can be replaced with $0 \le p_{11} \le P_1$. Before we proceed any further, we would first have to prove that W_{22} can be decoded at receiver 1. By Fano's inequality, we have $h(W_{22} \,|\, \mathbf{y}_2^n) \le n\epsilon_{4n}$.

Since W_{11} is the only message transmitted from transmitter 1, decoding and canceling out $\mathbf{x}_1^n$ will remove transmitter 1's contribution to $\mathbf{y}_1^n$. After this step, it is sufficient to show that $h(W_{22} \,|\, \mathbf{y}_1^n,\, \mathbf{x}_1^n) \le n\epsilon_{4n}$, since as $n \to \infty$, $\epsilon_{4n} \to 0$, proving that message W_{22} is decodable at receiver 1. For the degraded ZC of type II, by definition, we have the following Markov chain: $W_{22} \to \mathbf{x}_2^n \to (\mathbf{x}_1^n, \mathbf{y}_1^n) \to \mathbf{y}_2^n$. Using the data processing inequality, we conclude that

$$I(W_{22}\,;\, \mathbf{y}_1^n,\, \mathbf{x}_1^n) \;\ge\; I(W_{22}\,;\, \mathbf{y}_2^n) \quad (44)$$

from which it is straightforward to show that

$$h(W_{22} \,|\, \mathbf{y}_1^n,\, \mathbf{x}_1^n) \le n\epsilon_{4n}. \quad (45)$$

Finally, we bound the sum-rate of the remaining messages, $R_{11} + R_{12} + R_{22}$ as follows:

$$\begin{aligned} n(R_{11}+R_{12}+R_{22}) &\le I(W_{11}, W_{12}, W_{22}; \mathbf{y}_1^n, W_{21}) + n\epsilon_{5n} \\ &\le I(W_{11}, W_{12}, W_{22}; \mathbf{y}_1^n \,|\, W_{21}) + n\epsilon_{5n} \\ &\le h(\mathbf{y}_1^n \,|\, W_{21}) - h(\mathbf{n}_1^n) + n\epsilon_{5n}. \end{aligned}$$

Using (42) in the above equation, we get

$$R_{11}+R_{12}+R_{22} \;\le\; \log(1+|h_{11}|^2 p_{11} + |h_{12}|^2 P_2) + n\epsilon_{5n},$$

and as $n \to \infty$, $\epsilon_{5n} \to 0$ and we get the desired bound. This completes the proof.

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