On the Maximal Rate of Non-square STBCs from Complex Orthogonal Designs

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Abstract*— A Linear Processing Complex Orthogonal Design (LPCOD) is a $p \times n$ matrix $\mathcal{E}$, $(p \geq n)$ in k complex indeterminates $x_1, x_2, \cdots, x_k$ such that (i) the entries of $\mathcal{E}$ are complex linear combinations of $0, \pm x_i,\ \ i = 1, \cdots, k$ and their conjugates, (ii) $\mathcal{E}^H \mathcal{E} = \mathbf{D}$, where $\mathcal{E}^H$ is the Hermitian (conjugate transpose) of $\mathcal{E}$ and $\mathbf{D}$ is a diagonal matrix with the (i,i)-th diagonal element of the form $l_1^{(i)}|x_1|^2 + l_2^{(i)}|x_2|^2 + \cdots + l_k^{(i)}|x_k|^2$ where $l_j^{(i)}, i = 1, 2, \cdots, n,\ \ j = 1, 2, \cdots, k$ are strictly positive real numbers and the condition $l_1^{(i)} = l_2^{(i)} = \cdots = l_k^{(i)}$, called the equal-weights condition, holds for all values of i. For square designs it is known that whenever a LPCOD exists without the equal-weights condition satisfied then there exists another LPCOD with identical parameters with $l_1^{(i)} = l_2^{(i)} = \cdots = l_k^{(i)} = 1$. This implies that the maximum possible rate for square LPCODs without the equal-weights condition is the same as that of square LPCODs with equal-weights condition. In this paper, this result is extended to a subclass of non-square LPCODs. It is shown that, a set of sufficient conditions is identified such that whenever a non-square $(p > n)$ LPCOD satisfies these sufficient conditions and do not satisfy the equal-weights condition, then there exists another LPCOD with the same parameters n, k and p in the same complex indeterminates with $l_1^{(i)} = l_2^{(i)} = \cdots = l_k^{(i)} = 1$.

I. Introduction

Orthogonal designs [1], [2] have been extensively studied due to the fact that the space-time block codes (STBCs) obtained using these designs by way of letting the variables of the design to take values from different signal sets admit single-symbol maximum likelihood (ML) decodability [3]. These STBCs admit single-real-symbol decodability for regular QAM signal sets and single-complex-symbol decodability for arbitrary complex signal sets including PSK. Designs leading to STBCs admitting single-complex-symbol decodability have been studied in detail in [3], where it is shown that orthogonal designs constitute a proper subclass of such designs.

In this paper, we restrict ourselves to orthogonal designs covered by the following definition: A $[p, n, k]$ Generalized Linear Processing Complex Orthogonal Design (GLPCOD) [1], [2] is a $p \times n$ matrix $\mathcal{E}$ with entries that are arbitrary complex linear combinations of k complex indeterminates $x_1, x_2, \cdots, x_k$ and their complex conjugates,with $p \geq n$ such that

- $\mathcal{E}^H \mathcal{E} = \mathbf{D}$, where $\mathcal{E}^H$ is the Hermitian (conjugate transpose) of $\mathcal{E}$ and $\mathbf{D}$ is a diagonal matrix with the (i,i)-th diagonal element of the form
$$l_1^{(i)}|x_1|^2 + l_2^{(i)}|x_2|^2 + \cdots + l_k^{(i)}|x_k|^2,$$
where $l_j^{(i)}, i = 1, 2, \cdots, n,\ \ j = 1, 2, \cdots, k$ are strictly positive real numbers, and for all values of i,
$$l_1^{(i)} = l_2^{(i)} = \cdots = l_k^{(i)}. \quad (1)$$

The condition given by (1), known as the **equal-weights condition** [4], has been introduced in [2] as a correction to [1]. If k=n=p, then $\mathcal{E}$ is called a Linear Processing Complex Orthogonal Design (LPCOD). Furthermore, when the entries are only from $\{0, \pm x_1, \pm x_2, \cdots, \pm x_k\}$, their conjugates and multiples of $\mathbf{j}$, where $\mathbf{j} = \sqrt{-1}$, then $\mathcal{E}$ is called a Complex Orthogonal Design (COD). When the entries of $\mathcal{E}$ are real variables and real linear combinations of these variables, Generalized Linear Processing Real Orthogonal Designs (GLPROD), Linear Processing Real Orthogonal Design (LPROD) and Real Orthogonal Designs (ROD) are similarly defined.

The existence of Orthogonal Designs is of fundamental importance in the theory of Space-Time Block Codes [1]. In this regard, [1] presents four theorems (Theorems 3.4.1, 4.1.1, 5.4.1 and 5.5.1): Theorem 3.4.1 deals with RODs, Theorem 5.4.1 deals with CODs, and Theorems 4.1.1 and 5.5.1 deal with GLPROD and GLPCOD, respectively. All these theorems assume that equal-weights condition holds. In view of these theorems, as far as the existence of designs are concerned, one may, without any loss of generality, assume that a generalized linear processing real or complex orthogonal design satisfies $1 = l_1^{(i)} = l_2^{(i)} = \cdots = l_k^{(i)}$ given that $l_1^{(i)} = l_2^{(i)} = \cdots = l_k^{(i)}$.

Maximal rate of square orthogonal designs have been studied in [6] and that of non-square orthogonal designs have been studied in [7], [8]. In [4] it has been shown that the maximum rate of complex orthogonal STBCs with equal-weights conditions satisfied is not different from that of square complex orthogonal STBCs without equal-weight condition satisfied, by showing that, in case of square GLPCODs, Theorems 3.4.1 and 5.4.1 of [1] are valid without the equal-weights condition in the definition of GLPCODs. Notice that the number of variables k need not be equal to $n = p$. To be precise, the following theorem has been proved in [4].

Theorem 1: With the equal-weights condition removed from the definition of GLPCODs, an $n \times n$ square GLPCOD

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$\mathcal{E}_c$, in variables $x_1, \cdots, x_k$ exists iff there exists a GLPCOD $\mathcal{L}_c$ of same size and in the same variables such that

$$\mathcal{L}_c^H \mathcal{L}_c = (|x_1|^2 + \cdots + |x_k|^2)I.$$

A. Non-square design applications

Most studies on STBCs from orthogonal designs so far dealt with square designs, since they correspond to minimum delay codes for co-located multiple antenna coherent communication systems. However, non-square designs are important in several other important situations, some of which are:

1) In non-coherent MIMO systems with non-differential detection, non-square designs with $p = 2n$ lead to low decoding complexity STBCs [10].
2) Space-Time-Frequency codes can be viewed as non-square designs [11].
3) In distributed space-time coding for relay channels rectangular designs naturally appear [12].
4) In coherent co-located MIMO systems, for a specified number of transmit antennas, non-square designs can give much higher rate than the square designs [7].

B. Our contribution

In this paper, we identify a subclass of non-square LPCODs and prove that for code within this subclass, the maximum rate of non-square LPCODs without satisfying the equal-weights condition is the same as the maximal rate of non-square LPCODs with the equal-weights condition satisfied for identical set of the parameters p, n, and k and in the same set of variables. This is achieved by way of identifying a set of sufficient conditions such that whenever a LPCOD without satisfying the equal-weights condition satisfies these sufficient conditions then there exists another LPCOD with the same set of parameters and variables with equal-weights condition satisfied. Our proof also provides a method of obtaining the LPCODs with equal-weight condition from a LPCOD without equal-weight conditions, for the class of codes satisfying our set of sufficient conditions.

The remaining content is organized as follows: The main result of this paper is Theorem 5, which is proved in Section II. In Section III, it is shown that certain substitutions used in the proof of Theorem 5 can be used to obtain a design with identical parameters and variables satisfying the equal-weights condition from a non-square design not satisfying the equal-weights condition by way of illustration with two example designs from [9]. Concluding remarks regarding the extent of usefulness of the results of this paper constitute Section IV.

II. Non-square GLPCODs with maximal rate independent of equal-weights condition

In this section, we prove a generalization of Theorems 4.4.1 and 5.5.1 of [1]. We reproduce below Theorem 4.4.1 and 5.5.1 of [1] for quick reference. Note that these two theorems have been stated assuming the equal-weights condition to be part of the definition of GLPCODs.

Theorem 2 (Theorem 4.1.1 of [1]): A $p \times n$ generalized linear processing real orthogonal design, $\mathcal{E}$, in real variables $x_1, x_2, \cdots, x_k$ exists iff there exists a generalized linear processing real orthogonal design $\mathcal{G}$, in the same variables and of the same size such that

$$\mathcal{G}^T \mathcal{G} = (x_1^2 + x_2^2 + \cdots + x_k^2)I.$$

Theorem 3 (Theorem 5.5.1 of [1]): A $p \times n$ linear processing complex orthogonal design, $\mathcal{E}_c$, in complex variables $x_1, x_2, \cdots, x_n$ and their conjugates exists iff there exists a generalized linear processing complex orthogonal design $\mathcal{G}_c$, such that

$$\mathcal{G}_c^H \mathcal{G}_c = (|x_1|^2 + |x_2|^2 + \cdots + |x_k|^2)I.$$

Given a $p \times n$ GLPCOD $\mathcal{E}_c$ in k complex variables, by $\mathcal{E}_c^{(m)}$ we denote the m^{th} column of $\mathcal{E}_c$. Moreover, we denote the $2k$ real variables consisting of all the in-phase and quadrature components of the k complex variables by $y_i, \quad i = 1, 2, \cdots, 2k$. Then, we can express $\mathcal{E}_c^{(m)}$ as

$$\mathcal{E}_c^{(m)} = \sum_{i=1}^{2k} y_i C_i^{(m)}, \tag{2}$$

where, for all $m = 1, \cdots n$, $C_i^{(m)}$ is the $p \times 1$ complex weight vector associated with variable y_i in the m^{th} column of $\mathcal{E}_c$. Note that, $C_i^{(m)}$ is closely related to the $i-th$ column of the $m-$th "column vector representation of a LPCOD" as in [7]. We call $C_i^{(m)}$, the $i-$th *column-weight-vector* for the $m-$th column of the design.

The following theorem characterizes non-square LPCODs in terms of the corresponding set of column-weight-matrices of the design.

Theorem 4: Let $\mathcal{E}_c$ be a $p \times n$ GLPCOD in $2k$ real variables $y_i \quad i = 1, 2, \cdots, 2k$. Then,

$$\mathcal{E}_c^H \mathcal{E}_c = \sum_{i=1}^{2k} y_i^2 \mathcal{D}_i \tag{3}$$

where $\mathcal{D}_i$ is the $n \times n$ diagonal real matrix (with strictly positive entries) associated with the variable y_i if and only if the column-weight-vectors $C_i^{(m)}, \ i = 1, 2, \cdots, 2k$ and $m = 1, 2, \cdots, n$ satisfy the following four conditions:

$$C_i^{(m)H} C_i^{(m)} = \mathcal{D}_i^{(m,m)}, \tag{4}$$
$$i = 1, 2, \cdots, 2k, \ m = 1, 2, \cdots, n$$

where $\mathcal{D}_i^{(m,m)}$ is the $(m, m)^{th}$ entry of the diagonal matrix $\mathcal{D}_i$;

$$C_p^{(m)H} C_q^{(m)} + C_q^{(m)H} C_p^{(m)} = 0. \tag{5}$$
$$p, q = 1, 2, \cdots, 2k, \ p \neq q, \quad m = 1, 2, \cdots, n;$$

$$C_i^{(m)H} C_i^{(l)} = 0. \tag{6}$$
$$i = 1, 2, \cdots, 2k, \quad m, l = 1, 2, \cdots, n, \ m \neq l;$$

$$C_p^{(m)H} C_q^{(l)} + C_q^{(m)H} C_p^{(l)} = 0, \tag{7}$$
$$p, q = 1, 2, \cdots, 2k, \ p \neq q, \quad m, l = 1, 2, \cdots, n, \ m \neq l.$$

Proof: The proof is omitted due to space considerations. ■

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Next, we define a transformation that transforms the (p,n,k) GLPCOD $\mathcal{E}_c$ to another (p,n,k) GLPCOD $\mathcal{G}_c$ in the same variables, by defining the m^{th} column of $\mathcal{G}_c$, denoted by $\mathcal{G}_c^{(m)}$ to be,

$$\mathcal{G}_c^{(m)} = \sum_{i=1}^{2k} y_i E_i^{(m)} \tag{8}$$

where

$$E_i^{(m)} = (1/\sqrt{\mathcal{D}_i^{(m,m)}})C_i^{(m)}, \tag{9}$$
$$i = 1,2,\cdots,2k, \quad m = 1,2,\cdots,n.$$

where y_i is the i^{th} variable of both $\mathcal{G}_c$ and $\mathcal{E}_c$ and $C_i^{(m)}$ is the column-weight-vector of y_i for the m^{th} column of $\mathcal{E}_c$. The main result of the paper is Theorem 5 which gives a sufficient condition on the entries of the diagonal matrices $\mathcal{D}_i$ of the design $\mathcal{E}_c$, such that the GLPCOD $\mathcal{G}_c$ satisfies

$$\mathcal{G}_c^H \mathcal{G}_c = \sum_{i=1}^{2k} y_i^2 I \tag{10}$$

where I is the $n \times n$ identity matrix.

Theorem 5: If the diagonal matrices $\mathcal{D}_i$ corresponding to the design $\mathcal{E}_c$ satisfy the following condition then the GLPCOD $\mathcal{G}_c$ will satisfy (10): For each $i,j = 1,2,\cdots,2k, \ i \neq j$ and $m,l = 1,\cdots,n$, whenever $C_i^{(m)H}C_j^{(l)} \neq 0$ then $\mathcal{D}_i$ and $\mathcal{D}_j$ matrices satisfy

$$\mathcal{D}_i^{(m,m)}\mathcal{D}_j^{(l,l)} = \mathcal{D}_j^{(m,m)}\mathcal{D}_i^{(l,l)}. \tag{11}$$

Proof: For $\mathcal{G}_c$ to satisfy (10), the necessary and sufficient conditions for the column-weight-vectors are, from Theorem (4),

$$E_i^{(m)H}E_i^{(m)} = 1, \tag{12}$$
$$i = 1,2,\cdots,2k, \quad m = 1,2,\cdots,n;$$

$$E_i^{(m)H}E_i^{(l)} = 0, \tag{13}$$
$$i = 1,2,\cdots,2k, \quad l,m = 1,2,\cdots,n, \quad l \neq m;$$

$$E_i^{(m)H}E_j^{(l)} + E_j^{(m)H}E_i^{(l)} = 0, \tag{14}$$
$$i,j = 1,2,\cdots,2k, \ i \neq j, \quad l,m = 1,2,\cdots,n.$$

In (14) we have combined the two conditions for $E_i^{(m)}$ based upon (5) and (7) into one condition by removing the constraint $l \neq m$. Upon using the definition of $E_i^{(m)}$ as in (9), and given that the vectors $C_i^{(m)}$ satisfy the conditions in Theorem (4), we see that conditions (12) and (13) are indeed true.

Using the definition for $E_i^{(m)}$ as defined in (9), (14) can be re-written as

$$C_i^{(m)H}C_j^{(l)}/\sqrt{\mathcal{D}_i^{(m,m)}\mathcal{D}_j^{(l,l)}} = -C_j^{(m)H}C_i^{(l)}/\sqrt{\mathcal{D}_j^{(m,m)}\mathcal{D}_i^{(l,l)}}. \tag{15}$$

If $C_i^{(m)H}C_j^{(l)} = 0$, then the condition (14) holds true trivially. However if $C_i^{(m)H}C_j^{(l)} \neq 0$, but $\mathcal{D}_i^{(m,m)}\mathcal{D}_j^{(l,l)} = \mathcal{D}_j^{(m,m)}\mathcal{D}_i^{(l,l)}$, then also (14) hold true. Hence we have proved that if the conditions in (11) are met, then $\mathcal{G}_c$ indeed satisfies (10). ■

III. Designs with equal-weights condition from designs without it

In [9], Su and Xia present a $[11,5,7]$, rate-$7/11$ design and a $[30,6,18]$, rate-$3/5$ design that do not satisfy the equal-weights condition. In this section, we illustrate the construction of designs with identical parameters as these codes in the same set of variables, but satisfying the equal-weights condition. We achieve this by making use of the transformations given by (9) on the set of column-weight-vectors of the codes of Su and Xia.

The rate-$7/11$ code of [9] for 5 transmit antennas is given by

$$\mathcal{E}_1 = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & x_4 \\ -x_2^* & x_1^* & 0 & x_3 & x_5 \\ x_3^* & 0 & -x_1^* & x_2 & x_6 \\ 0 & x_3^* & -x_2^* & -x_1 & x_7 \\ x_4^* & 0 & 0 & -x_7^* & -x_1^* \\ 0 & x_4^* & 0 & x_6^* & -x_2^* \\ 0 & 0 & x_4^* & x_5^* & -x_3^* \\ 0 & -x_5^* & x_6^* & 0 & x_1 \\ x_5^* & 0 & x_7^* & 0 & x_2 \\ -x_6^* & -x_7^* & 0 & 0 & x_3 \\ x_7 & -x_6 & -x_5 & x_4 & 0 \end{bmatrix}.$$

With the relabeling of the design variables $x_k = x_{kI} + jx_{kQ}$ as $y_{2k-1} = x_{kI}$ and $y_{2k} = x_{kQ}$ for $k = 1,2,\cdots,7$, the matrices $\mathcal{D}_i, \ i = 1,2,\cdots,14$ for the code $\mathcal{E}_1$ are

$$\mathcal{D}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \text{ for } i = 1,2,\cdots,6$$

and $\mathcal{D}_i = I_5$ for $i = 7,8,\cdots,14$. It can be verified that these matrices satisfy the condition given by (11). Using the transform given in (9), the corresponding code with equal-weights condition satisfied is

$$\mathcal{E}_{1,EW} = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & x_4 \\ -x_2^* & x_1^* & 0 & x_3 & x_5 \\ x_3^* & 0 & -x_1^* & x_2 & x_6 \\ 0 & x_3^* & -x_2^* & -x_1 & x_7 \\ x_4^* & 0 & 0 & -x_7^* & \frac{-x_1^*}{\sqrt{2}} \\ 0 & x_4^* & 0 & x_6^* & \frac{-x_2^*}{\sqrt{2}} \\ 0 & 0 & x_4^* & x_5^* & \frac{-x_3^*}{\sqrt{2}} \\ 0 & -x_5^* & x_6^* & 0 & \frac{x_1}{\sqrt{2}} \\ x_5^* & 0 & x_7^* & 0 & \frac{x_2}{\sqrt{2}} \\ -x_6^* & -x_7^* & 0 & 0 & \frac{x_3}{\sqrt{2}} \\ x_7 & -x_6 & -x_5 & x_4 & 0 \end{bmatrix}. \tag{16}$$

Notice that for the design $\mathcal{E}_{1,EW}$, all the matrices $\mathcal{D}_i, \ \ i = 1,2,\cdots,14$ are identity matrices.

The second code of [9] for 6 antennas of rate-$3/5$, is shown below:

$$\mathcal{E}_2 = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & x_4 & x_8 \\ -x_2^* & x_1^* & 0 & x_3 & x_5 & x_9 \\ x_3^* & 0 & -x_1^* & x_2 & x_6 & x_{10} \\ 0 & x_3^* & -x_2^* & -x_1 & x_7 & x_{11} \\ x_4^* & 0 & 0 & -x_7^* & -x_1^* & x_{12} \\ 0 & x_4^* & 0 & x_6^* & -x_2^* & x_{13} \\ 0 & 0 & x_4^* & x_5^* & -x_3^* & x_{14} \\ 0 & x_5^* & -x_6^* & 0 & -x_1 & x_{15} \\ x_5^* & 0 & x_7^* & 0 & x_2 & x_{16} \\ x_6^* & x_7^* & 0 & 0 & -x_3 & x_{17} \\ x_7 & -x_6 & -x_5 & x_4 & 0 & x_{18} \\ x_8^* & 0 & 0 & -x_{11}* & -x_{15}^* & -x_1^* \\ 0 & x_8^* & 0 & x_{10}^* & x_{16}^* & -x_2^* \\ 0 & 0 & x_8^* & x_9^* & -x_{17}^* & -x_3^* \\ 0 & 0 & 0 & x_{18}^* & x_8^* & -x_4^* \\ 0 & 0 & -x_{18}^* & 0 & x_9^* & -x_5^* \\ 0 & -x_{18}^* & 0 & 0 & x_{10}^* & -x_6^* \\ x_{18}^* & 0 & 0 & 0 & x_{11}^* & -x_7^* \\ 0 & -x_9^* & x_{10}^* & 0 & x_{12}^* & x_1 \\ x_9^* & 0 & x_{11}^* & 0 & x_{13}^* & x_2 \\ -x_{10}^* & -x_{11}^* & 0 & 0 & x_{14}^* & x_3 \\ -x_{12}^* & -x_{13}^* & -x_{14}^* & 0 & 0 & x_4 \\ -x_{16}^* & -x_{15}^* & 0 & -x_{14}^* & 0 & x_5 \\ -x_{17}^* & 0 & x_{15}^* & -x_{13}^* & 0 & x_6 \\ 0 & x_{17}^* & -x_{16}^* & x_{12}^* & 0 & x_7 \\ 0 & x_{14} & -x_{13} & -x_{15} & x_{11} & 0 \\ x_{14} & 0 & -x_{12} & -x_{16} & x_{10} & 0 \\ -x_{13} & x_{12} & 0 & x_{17} & x_9 & 0 \\ x_{15} & -x_{16} & x_{17} & 0 & x_8 & 0 \\ -x_{11} & x_{10} & x_9 & -x_8 & x_{18} & 0 \end{bmatrix}.$$

With the relabeling of the design variables $x_k = x_{kI} + jx_{kQ}$ as $y_{2k-1} = x_{kI}$ and $y_{2k} = x_{kQ}$ for $k = 1, 2, \cdots, 18$, that this code does not satisfy the equal-weights condition is clear from the following $\mathcal{D}_i, \ i = 1, 2, \cdots, 36$, matrices:

$$\mathcal{D}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \text{ for } \ i = 1, 2, \cdots, 6;$$

$$\mathcal{D}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \text{ for } \ i = 7, 8, \cdots, 14;$$

$$\mathcal{D}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ for } \ i = 15, 16, \cdots, 22,$$

and $\mathcal{D}_i = I_6$ for $i = 23, 24, \cdots, 36$.

It is easily checked that these matrices satisfy the condition given by (11) and using the transform given in (9), the corresponding code which satisfies the equal-weights condition is

$$\mathcal{E}_{2,EW} = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & x_4 & x_8 \\ -x_2^* & x_1^* & 0 & x_3 & x_5 & x_9 \\ x_3^* & 0 & -x_1^* & x_2 & x_6 & x_{10} \\ 0 & x_3^* & -x_2^* & -x_1 & x_7 & x_{11} \\ x_4^* & 0 & 0 & -x_7^* & \frac{-x_1^*}{\sqrt{2}} & x_{12} \\ 0 & x_4^* & 0 & x_6^* & \frac{-x_2^*}{\sqrt{2}} & x_{13} \\ 0 & 0 & x_4^* & x_5^* & \frac{-x_3^*}{\sqrt{2}} & x_{14} \\ 0 & x_5^* & -x_6^* & 0 & \frac{-x_1}{\sqrt{2}} & x_{15} \\ x_5^* & 0 & x_7^* & 0 & \frac{x_2}{\sqrt{2}} & x_{16} \\ x_6^* & x_7^* & 0 & 0 & \frac{-x_3}{\sqrt{2}} & x_{17} \\ x_7 & -x_6 & -x_5 & x_4 & 0 & x_{18} \\ x_8^* & 0 & 0 & -x_{11}* & -x_{15}^* & \frac{-x_1^*}{\sqrt{2}} \\ 0 & x_8^* & 0 & x_{10}^* & x_{16}^* & \frac{-x_2^*}{\sqrt{2}} \\ 0 & 0 & x_8^* & x_9^* & -x_{17}^* & \frac{-x_3^*}{\sqrt{2}} \\ 0 & 0 & 0 & x_{18}^* & \frac{x_8^*}{\sqrt{2}} & \frac{-x_4^*}{\sqrt{2}} \\ 0 & 0 & -x_{18}^* & 0 & \frac{x_9^*}{\sqrt{2}} & \frac{-x_5^*}{\sqrt{2}} \\ 0 & -x_{18}^* & 0 & 0 & \frac{x_{10}^*}{\sqrt{2}} & \frac{-x_6^*}{\sqrt{2}} \\ x_{18}^* & 0 & 0 & 0 & \frac{x_{11}^*}{\sqrt{2}} & \frac{-x_7^*}{\sqrt{2}} \\ 0 & -x_9^* & x_{10}^* & 0 & x_{12}^* & \frac{x_1}{\sqrt{2}} \\ x_9^* & 0 & x_{11}^* & 0 & x_{13}^* & \frac{x_2}{\sqrt{2}} \\ -x_{10}^* & -x_{11}^* & 0 & 0 & x_{14}^* & \frac{x_3}{\sqrt{2}} \\ -x_{12}^* & -x_{13}^* & -x_{14}^* & 0 & 0 & \frac{x_4}{\sqrt{2}} \\ -x_{16}^* & -x_{15}^* & 0 & -x_{14}^* & 0 & \frac{x_5}{\sqrt{2}} \\ -x_{17}^* & 0 & x_{15}^* & -x_{13}^* & 0 & \frac{x_6}{\sqrt{2}} \\ 0 & x_{17}^* & -x_{16}^* & x_{12}^* & 0 & \frac{x_7}{\sqrt{2}} \\ 0 & x_{14} & -x_{13} & -x_{15} & \frac{x_{11}}{\sqrt{2}} & 0 \\ x_{14} & 0 & -x_{12} & -x_{16} & \frac{x_{10}}{\sqrt{2}} & 0 \\ -x_{13} & x_{12} & 0 & x_{17} & \frac{x_9}{\sqrt{2}} & 0 \\ x_{15} & -x_{16} & x_{17} & 0 & \frac{x_8}{\sqrt{2}} & 0 \\ -x_{11} & x_{10} & x_9 & -x_8 & x_{18} & 0 \end{bmatrix}.$$

As was the case with the code $\mathcal{E}_{1,EW}$, notice that for the design $\mathcal{E}_{2,EW}$ also, all the matrices $\mathcal{D}_i, \quad i = 1, 2, \cdots, 14$ are identity matrices.

IV. Discussion

For a given set of parameters p, n, k, complex orthogonal design satisfying the equal-weights condition is not unique. There can be two complex orthogonal designs satisfying the equal-weights condition in the same set of variables as shown below for 4 antennas. The well known square COD for 4 transmit antenna [1]

$$\begin{bmatrix} x_1 & x_2 & -x_3^* & 0 \\ -x_2^* & x_1^* & 0 & -x_3^* \\ x_3 & 0 & x_1^* & -x_2 \\ 0 & x_3 & x_2^* & x_1 \end{bmatrix},$$

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and the following code given in [13], [1] obtained from Amicable Orthogonal Designs

$$\begin{bmatrix} x_1 & x_2 & \frac{x_3^*}{\sqrt{2}} & \frac{x_3^*}{\sqrt{2}} \\ -x_2^* & x_1^* & \frac{x_3^*}{\sqrt{2}} & \frac{-x_3^*}{\sqrt{2}} \\ \frac{x_3^*}{\sqrt{2}} & \frac{x_3^*}{\sqrt{2}} & \frac{(-x_1-x_1^*+x_2-x_2^*)}{2} & \frac{(+x_1-x_1^*-x_2-x_2^*)}{2} \\ \frac{x_3^*}{\sqrt{2}} & \frac{-x_3^*}{\sqrt{2}} & \frac{(x_1-x_1^*+x_2+x_2^*)}{2} & \frac{(x_1+x_1^*+x_2-x_2^*)}{2} \end{bmatrix},$$

have the same code parameters and the same set of variables and both satisfy the equal-weights condition. Similar codes can be obtained for 8 antennas as shown below: The square COD given in [1] for 8 antennas

$$G = \begin{bmatrix} x_1 & -x_2^* & -x_3^* & 0 & -x_4^* & 0 & 0 & 0 \\ x_2 & x_1^* & 0 & -x_3^* & 0 & -x_4^* & 0 & 0 \\ x_3 & 0 & x_1^* & x_2^* & 0 & 0 & -x_4^* & 0 \\ 0 & x_3 & -x_2 & x_1 & 0 & 0 & 0 & -x_4^* \\ x_4 & 0 & 0 & 0 & x_1^* & x_2^* & x_3^* & 0 \\ 0 & x_4 & 0 & 0 & -x_2 & x_1 & 0 & x_3^* \\ 0 & 0 & x_4 & 0 & -x_3 & 0 & x_1 & -x_2^* \\ 0 & 0 & 0 & x_4 & 0 & -x_3 & x_2 & x_1^* \end{bmatrix}$$

contains 50 per cent of the entries as zeros. But, Yuen et al [14] have constructed a rate $1/2$ square COD, $\frac{G_Y}{\sqrt{2}}$, of size 8 with no zeros in the design matrix using Amicable Complex Orthogonal Design (ACOD)[13], where G_Y is given by

$$\begin{bmatrix} x_1^* & x_1^* & x_2 & -x_2 & x_3 & -x_3 & x_4 & -x_4 \\ jx_1 & -jx_1 & jx_2^* & jx_2^* & jx_3^* & jx_3^* & jx_4^* & jx_4^* \\ -x_2 & x_2 & x_1^* & x_1^* & x_4^* & -x_4^* & -x_3^* & x_3^* \\ -jx_2^* & -jx_2^* & jx_1 & -jx_1 & jx_4 & jx_4 & -jx_3 & -jx_3 \\ -x_3 & x_3 & -x_4^* & x_4^* & x_1^* & x_1^* & x_2^* & -x_2^* \\ -jx_3^* & -jx_3^* & -jx_4 & -jx_4 & jx_1 & -jx_1 & jx_2 & jx_2 \\ -x_4 & x_4 & x_3^* & -x_3^* & -x_2^* & x_2^* & x_1^* & x_1^* \\ -jx_4^* & -jx_4^* & jx_3 & jx_3 & -jx_2 & -jx_2 & jx_1 & -jx_1 \end{bmatrix},$$

which has no zeros in the matrix. The different number of zeros in the designs indicates that their performance under peak power constraints will be different. This aspect has been studied in detail in [5] for square designs and it will be interesting to extend this for non-square designs.

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